

GENERAL SPINOR STRUCTURES ON QUANTUM SPACES

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ABSTRACT. A general theory of quantum spinor structures on quantum spaces is presented, within the conceptual framework of the formalism of quantum principal bundles. Quantum analogs of all basic objects of the classical theory are constructed and analyzed. This includes Laplace and Dirac operators, quantum versions of Clifford and spinor bundles, a Hodge $*$ -operator, appropriate integration operators, and mutual relations of these objects. We also present a self-contained formalism of braided Clifford algebras. Quantum phenomena appearing in the theory are discussed, including a very interesting example of the Dirac operator associated to a quantum Hopf fibration.

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1. INTRODUCTION

Classical theory of spinors is a cornerstone of important bridges that interconnect worlds of geometry, algebra and physics.

Just as an illustration—let us recall that using spinor structures, it is possible to address various fundamental topics of Riemannian geometry in an elegant and effective way. The Dirac operator is intrinsically related with a general index theory of elliptic operators. Spin bundles provide powerful tools for the study of gauge field theories with fermions.

Classical geometry is a very special case of quantum (\Leftrightarrow non-commutative) geometry, which incorporates many ideas of quantum physics into the world of geometry. And if we try to understand the structure of physical space-time at very small scales, it becomes clear that classical concepts of the space-time continuum loose their validity.

It is natural to look for a quantum version of the spinor theory, hoping that it would be at least as interesting as its classical counterpart.

The aim of this study is to present a general theory of spinor structures over quantum spaces, in the spirit of non-commutative differential geometry [2]. The main conceptual framework for our considerations is the theory of quantum principal bundles [3, 4], where quantum groups play the role of structure groups and general quantum spaces play the role of base manifolds.

The formalism presented here could be used as a possible tool for developing a theory of fermions over a quantum space-time, which would be appropriate at the level of ultra-small distances characterized by the Planck length. Our formulation fulfills various conditions proposed in a general axiomatic framework [2]. However, the formulation of [2] does not require quantum principal bundles as the basic underlying structure. Further, some key conditions of [2] are broken in our formalism. This includes the spectral asymptotics of the quantum Dirac operator, which in our case could be very different from the classical behavior. In particular, our constructions are not compatible with the Dixmier trace.

As far as pure geometry is concerned, our constructions provide a coherent framework for the formulation of quantum elliptic operators [10] and the study of the corresponding index theorems.

The results of this paper include as a special case the formalism of spin structures studied in [6], where quantum spin bundles with classical structure groups were considered (and it was assumed that the differential calculus over the structure group is classical, too).

The paper is organized as follows.

In the next section we are going to introduce basic structural elements of quantum Riemannian geometry. We shall use the general theory of frame structures on quantum principal bundles [5], in order to develop the idea of a quantum space equipped with a metric. In particular, we shall explain how to construct a graded \ast -algebra \mathfrak{hor}_P representing horizontal forms, starting from ‘abstract coordinate 1-forms’ and a quantum principal bundle P . In accordance with [5], the space of abstract coordinate 1-forms \mathbb{V} will be defined as the left-invariant part of a bicovariant bimodule Ψ over the structure quantum group G . The group G acts on \mathbb{V} by ‘orthogonal transformations’. The space \mathbb{V} carries a very interesting geometrical structure, and in particular [22] there exists a canonical braid operator $\tau: \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$ playing the role of the transposition map. Furthermore, we

shall introduce abstract Levi-Civita connections. In a certain sense, these objects contain the whole geometrical information about quantum frame structures.

This is the most subtle part of the formalism, as it requires to introduce carefully a number of non-trivial conditions on the base space M , the structure quantum group G and the bundle P .

After presenting the main algebraic setup, and reviewing basic ideas of [5], we shall consider certain special conditions which will further justify our geometrical interpretation of M —as a quantum space equipped with a metric.

This includes analytic conditions—the existence of the appropriate C^* -algebraic completions of both the base space and the bundle $*$ -algebras, as well as the existence of a suitable ‘homogeneous’ measure on the base space M . Combining this measure with the integration along the fibers of P we shall construct a natural measure on the bundle.

One of the main purely algebraic extra conditions will be the existence of a ‘volume element’ in the algebra of coordinate horizontal forms. With the help of the volume element and the measure on P , it will be possible to construct the integration map $\int_P: \mathfrak{hor}_P \rightarrow \mathbb{C}$.

A quantum version of the Euclidean structure on \mathbb{V} will be represented by a metric form $g: \mathbb{V} \otimes \mathbb{V} \rightarrow \Sigma$, where Σ is a $*$ -algebra of *abstract metric tensor coefficients*. In general $\Sigma \neq \mathbb{C}$, and there exist deep reasons why it is necessary to assume that components of the metric generate a non-commutative $*$ -algebra.

After discussing the integration maps, we shall present the construction of the quantum Hodge $*$ -operator, with the help of which it will be possible to express elegantly a canonical scalar product of horizontal forms, along the lines of classical geometry. Next, we shall introduce the coderivative operator, the Laplace operator and analyze their properties. Explicit coordinate formulas for the Laplacian and the adjoint derivative will be derived, too.

In Section 3 we shall introduce the concept of a quantum spinor structure, in a complete analogy with classical geometry. These structures will be defined as certain ‘covering bundles’ of the ‘true orthonormal frame bundles’. Their structure group will be a kind of a quantum spin group. We shall proceed by introducing quantum versions of the Clifford bundle algebra and the associated spinor bundle.

Section 4 is devoted to the quantum Dirac operator. This operator acts in the quantum spinor bundle, and will be defined in a complete analogy with the classical geometry. We shall study its properties, including the relations with other important objects of the game. In particular, a quantum generalization of the Lichnerowicz formula [12] will be derived. We refer to [15] for a general diagrammatic braided-algebraic foundation of Dirac operator.

In Section 5 some concluding remarks and observations are made. We shall briefly consider the case of arbitrary metric connections (with a possibly non-vanishing torsion), and we shall also stress specific quantum phenomena appearing in the formalism. As a very instructive example, we shall sketch [13] the construction of the basic objects of the game in the case of the quantum 2-sphere [16] equipped with a canonical spin structure coming from the quantum Hopf fibering. It turns out that in the quantum case the spectrum of the Dirac operator radically differs from the classical situation. The structure group for the Hopf fibering is the classical $U(1)$ however the differential calculus over it will be *quantum*.

The paper ends with three appendices.

In the first one, we have included elementary informations about the C^* -algebraic completions of the $*$ -algebras \mathcal{V} and \mathcal{B} representing the base space and the bundle.

A natural GNS-type construction is sketched. This construction gives us a natural Hilbert space realization, associated to the vertical integration map, of the bundle $*$ -algebra \mathcal{B} .

In the second appendix, we present in detail the main construction of quantum Clifford algebras, associated to general braid operators. Conceptually, we shall follow [11]. Our Clifford algebras will be understood as deformations of the appropriate braided exterior algebras. As already mentioned, in contrast to [11] we shall assume here that the *metric components* generate a possibly non-commutative $*$ -algebra Σ (instead of just being complex numbers). A special attention will be given to a discussion of various subtle properties related to the above mentioned non-commutativity of metric coefficients.

In the last appendix we have collected important properties and definitions related to the concept of *associated vector bundles*. In accordance with [9, 4] these bundles are defined as \mathcal{V} -bimodules consisting of intertwiners between finite-dimensional representations of the structure group G and the right action of G on the bundle/horizontal forms. It is also explained how to introduce a natural scalar product in the associated vector bundles, and how our fundamental operators (as Laplacian, Hodge- $*$ and Dirac) naturally act in the intertwiner spaces.

Finally, a technical remark concerning $*$ -structures. Throughout this paper, $*$ -structures on graded algebras will be understood in two ways, depending on the context: as graded-antimultiplicative $*$ -involutions, when dealing with quantum differential forms and exterior algebras, or simply as antimultiplicative $*$ -involutions when dealing with braided Clifford algebras and bundles.

Acknowledgements

I am very grateful to Prof Zbigniew Oziewicz for kindly inviting me to take part in the Ixtapa Conference, and for numerous very interesting and fruitful discussions we have had on various occasions.

2. QUANTUM RIEMANNIAN GEOMETRY

2.1. Basic Concepts

In this section we shall establish the principal geometrical settings for our considerations—quantum Riemannian geometry. This will be done in two steps.

At first, we shall recall the definition and basic properties of quantum principal bundles [3, 4] and frame structures [5] on them. Quantum frame structures allow us to incorporate into the non-commutative context a fundamental concept of coordinate 1-forms, establishing a coherent framework for the study of geometrical structures on quantum spaces, in the spirit of classical theory. This level is sufficient to introduce general metric connections (together with Levi-Civita connections), covariant derivative, curvature and torsion operators.

Secondly, in order to focus on ‘true metric spaces’, we shall introduce some specific analytical properties (that complete our geometrical picture). In particular, we shall introduce the integration operators for both the frame bundle and the base, the Laplace operator, the adjoint differential and the Hodge $*$ -operator, and prove a couple of important, yet elementary properties.

Let G be a compact matrix quantum group [20, 21], formally represented by a C^* -algebra A and a unital $*$ -homomorphism $\phi: A \rightarrow A \otimes A$ satisfying

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A \otimes A \\ \phi \downarrow & & \downarrow \text{id} \otimes \phi \\ A \otimes A & \xrightarrow{\phi \otimes \text{id}} & A \otimes A \otimes A \end{array} \quad A \otimes A = \overline{\text{lin}\{a\phi(b)\}} = \overline{\text{lin}\{\phi(a)b\}}.$$

The elements of A are interpretable as ‘continuous functions’ over the quantum space G . The group structure is given by the coproduct map ϕ . Let $\mathcal{A} \subseteq A$ be an everywhere dense $*$ -subalgebra corresponding to polynomial functions on G . We have $\phi(\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{A}$ and \mathcal{A} is actually a Hopf $*$ -algebra. We shall denote by $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ and $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$ the antipode and the counit map respectively.

Let M be a compact quantum space represented by a $*$ -algebra \mathcal{V} and let $P = (\mathcal{B}, i, F)$ be a quantum principal G -bundle over M . By definition [4], this means that \mathcal{B} is a $*$ -algebra, $i: \mathcal{V} \rightarrow \mathcal{B}$ is a $*$ -monomorphism, and $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ is a counital $*$ -homomorphism such that the following properties hold:

(i)–*The action property.* The diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{F} & \mathcal{B} \otimes \mathcal{A} \\ F \downarrow & & \downarrow \text{id} \otimes \phi \\ \mathcal{B} \otimes \mathcal{A} & \xrightarrow{F \otimes \text{id}} & \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{A} \end{array}$$

is commutative.

(ii)–*The ‘orbit space’ identification.* We have

$$i(\mathcal{V}) = \{b \in \mathcal{B} \mid F(b) = b \otimes 1\}.$$

(iii)–*The freeness condition.* A linear map $X: \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ given by

$$X(q \otimes b) = qF(b)$$

is surjective.

It is important to mention that if the map X is surjective then it will be automatically injective, so that X is actually bijective [8]. Thus, the freeness condition introduces us naturally into the algebraic framework of Hopf-Galois extensions [18].

As we shall see later, in a special case of the frame structures representing ‘vanilla’ quantum Riemannian manifolds, the above freeness condition will be satisfied *automatically*.

Now we are going to introduce, following [5], the concept of a quantum frame structure.

Let Ψ be a bicovariant [22] bimodule over G . The corresponding left and right co/action maps will be denoted by $\ell_{\Psi}: \Psi \rightarrow \mathcal{A} \otimes \Psi$ and $\varrho_{\Psi}: \Psi \rightarrow \Psi \otimes \mathcal{A}$ respectively. Let $\mathbb{V} = \Psi_{inv}$ be the corresponding left-invariant part. There exists a natural identification $\Psi \leftrightarrow \mathcal{A} \otimes \mathbb{V}$ of left \mathcal{A} -modules. The structure of Ψ is encoded in the restricted right action $\varkappa = (\varrho_{\Psi}|_{\mathbb{V}}): \mathbb{V} \rightarrow \mathbb{V} \otimes \mathcal{A}$ and a natural right \mathcal{A} -module structure \circ on \mathbb{V} , given by $\vartheta \circ a = \kappa(a^{(1)})\vartheta a^{(2)}$. If Ψ is $*$ -covariant then the space \mathbb{V}

is $*$ -invariant. We have the following interesting compatibility conditions between $*$, \circ and \varkappa :

$$\begin{aligned}\varkappa* &= (* \otimes *)\varkappa & (\theta \circ a)^* &= \theta^* \circ \kappa(a)^* \\ \varkappa(\theta \circ a) &= \sum_k (\theta_k \circ a^{(2)}) \otimes \kappa(a^{(1)}) c_k a^{(3)},\end{aligned}$$

where $\sum_k \theta_k \otimes c_k = \varkappa(\theta)$.

We shall assume that an auxiliary \varkappa -invariant scalar product (\cdot) is defined on \mathbb{V} . However, for the purposes of our main considerations, the central role will be played by a *noncommutative scalar product* in \mathbb{V} , taking its values in an appropriate $*$ -algebra Σ , generated by ‘abstract metric tensor coefficients’.

Let $\tau: \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$ be the canonical braid operator [22] associated to Ψ . It is computed in terms of \varkappa and \circ , as

$$\tau(\eta \otimes \vartheta) = \sum_k \vartheta_k \otimes (\eta \circ c_k).$$

Let \mathbb{V}^\wedge be the corresponding τ -exterior algebra, obtained from \mathbb{V}^\otimes by factorizing through the space of quadratic relations $\text{im}(I + \tau)$. At this point it is natural to assume that $\ker(I + \tau) \neq \{0\}$. This ensures the non-triviality of the higher-order part of \mathbb{V}^\wedge .

In what follows, the algebras \mathbb{V}^\otimes and \mathbb{V}^\wedge will be equipped with the induced $\circ, *$ and \varkappa -structures (these induced structures will be denoted by the same symbols). The extended structures are constructed by postulating

$$\begin{aligned}\varkappa(\vartheta \eta) &= \varkappa(\vartheta) \varkappa(\eta) & \varkappa(1) &= 1 \otimes 1 \\ (\vartheta \eta)^* &= (-)^{\partial \vartheta \partial \eta} \eta^* \vartheta^* \\ (\vartheta \eta) \circ a &= (\vartheta \circ a^{(1)}) (\eta \circ a^{(2)}) & 1 \circ a &= \epsilon(a) 1.\end{aligned}$$

The following identities express mutual compatibility between τ and the maps $*$, \circ and \varkappa :

$$\begin{array}{ccc}\mathbb{V} \otimes \mathbb{V} & \xrightarrow{\varkappa} & \mathbb{V} \otimes \mathbb{V} \otimes \mathcal{A} \\ \tau \downarrow & & \downarrow \tau \otimes \text{id} \\ \mathbb{V} \otimes \mathbb{V} & \xrightarrow{\varkappa} & \mathbb{V} \otimes \mathbb{V} \otimes \mathcal{A} \\ & \varkappa & \\ & \tau* = *\tau^{-1} & \\ & \tau(\psi \circ a) = \tau(\psi) \circ a. & \end{array}$$

Let us denote by $C_\varkappa: \mathbb{V} \rightarrow \mathbb{V}$ the canonical intertwiner between \varkappa and its second contragradient \varkappa^{cc} . By construction, this map is positive and satisfies

$$\varkappa C_\varkappa = (C_\varkappa \otimes \kappa^2) \varkappa \quad \text{tr}(C_\varkappa) = \text{tr}(C_\varkappa^{-1}).$$

We shall assume that the scalar product on \mathbb{V} is such that

$$(2.1) \quad (x^*, y^*) = (y, C_\varkappa x) \quad \forall x, y \in \mathbb{V}.$$

To put it another way around, we can *define* C_\varkappa by the above formula. Let us observe that this formula implies

$$(2.2) \quad *C_\varkappa* = C_\varkappa^{-1} \quad C_\varkappa = [*]^\dagger *.$$

The operator C_{\varkappa} is associated to the modular properties of the Haar measure [20]. Polary decomposing the map $*$: $\mathbb{V} \rightarrow \mathbb{V}$ we obtain

$$(2.3) \quad * = J_{\varkappa} C_{\varkappa}^{1/2} = C_{\varkappa}^{-1/2} J_{\varkappa},$$

where $J_{\varkappa}: \mathbb{V} \rightarrow \mathbb{V}$ is an antiunitary involution (in other words $J_{\varkappa} = J_{\varkappa}^{\dagger} = J_{\varkappa}^{-1}$).

When dealing with various ‘coordinate expressions’, we shall use a fixed basis $\{\theta_1, \dots, \theta_d\}$ in \mathbb{V} . We shall assume that these vectors satisfy

$$(\theta_i, \theta_j) = \delta_{ij} \quad J_{\varkappa}(\theta_i) = \theta_i.$$

In this basis the representation $\varkappa: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathcal{A}$ is given by a unitary matrix $[\varkappa_{ij}]$, so that

$$\varkappa(\theta_i) = \sum_j \theta_j \otimes \varkappa_{ji} \quad C_{\varkappa}^{1/2}[\varkappa]C_{\varkappa}^{-1/2} = [\overline{\varkappa}].$$

It is easy to see that the matrix $[C_{\varkappa}^{1/2}]_{ij} = (\theta_i, C_{\varkappa}^{1/2}\theta_j)$ is orthogonal.

As mentioned in the introduction, a ‘quantum Euclidean’ space structure on \mathbb{V} will be specified by an appropriate quadratic form $g: \mathbb{V} \otimes \mathbb{V} \rightarrow \Sigma$, playing the role of the metric, where Σ is a $*$ -algebra generated by matrix elements of g and g^{-1} , together with a new regular braid operator $\sigma: \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$ expressing the twisting properties of Σ and \mathbb{V} . This reflects a fundamental property of our theory, the non-commutativity of the metric tensor coefficients and the braided nature of Σ . In general $\sigma \neq \tau$.

The full set of properties involving g, σ and Σ is discussed in Appendix B. Accordingly, we shall also assume that $\{\sigma, \tau\}$ form a pair of coupled braid operators, so that the following natural identifications

$$(2.4) \quad \left\{ \tau\text{-antisymmetric } n\text{-tensors} \right\} \leftrightarrow \text{im}(A_{\sigma}^n)$$

hold for each $n \geq 2$. Here $A_{\sigma}^n: \mathbb{V}^{\otimes n} \rightarrow \mathbb{V}^{\otimes n}$ are the braided σ -antisymmetrizers. By definition, τ -antisymmetric tensors change the sign under all the corresponding τ -transpositions. Our τ -exterior algebra will be realizable as follows

$$\mathbb{V}^{\wedge} \leftrightarrow \mathbb{V}^{\otimes} / \ker(A_{\sigma}) \leftrightarrow \text{im}(A_{\sigma}).$$

These identifications are constructed with the help of A_{σ} . It is worth mentioning that the grading and the $*$ -structure are preserved in this picture. The maps C_{\varkappa} and J_{\varkappa} will be extended (by multiplicativity/anti) to \mathbb{V}^{\otimes} . The space \mathbb{V}^{\wedge} is invariant under the action of these maps. As discussed in Appendix B, there is a natural twisted tensor product of algebras \mathbb{V}^{\wedge} and Σ , and in such a way we obtain an extended braided exterior algebra $\mathbb{V}_{\Sigma}^{\wedge}$. This is a braided exterior algebra built over \mathbb{V}_{Σ} with the help of the extended braid $\sigma: \mathbb{V}_{\Sigma} \otimes_{\Sigma} \mathbb{V}_{\Sigma} \rightarrow \mathbb{V}_{\Sigma} \otimes_{\Sigma} \mathbb{V}_{\Sigma}$.

By definition, a *frame structure* on a quantum principal bundle P is given by a special graded $*$ -algebra \mathfrak{hor}_P , equipped with a first-order hermitian antiderivation $\nabla: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$. The algebra \mathfrak{hor}_P is defined as $\mathfrak{hor}_P \leftrightarrow \mathcal{B} \otimes \mathbb{V}^{\wedge}$ at the level of vector spaces, while the product and the $*$ -structure are given by formulae

$$(q \otimes \vartheta)(b \otimes \eta) = \sum_k q b_k \otimes (\vartheta \circ c_k) \eta$$

$$(b \otimes \vartheta)^* = \sum_k b_k^* \otimes (\vartheta^* \circ c_k^*)$$

where $\sum_k b_k \otimes c_k = F(b)$. The elements of \mathfrak{hor}_P are interpreted as ‘quantum horizontal forms’. The elements of \mathbb{V}^\wedge , viewed in the framework of \mathfrak{hor}_P , are interpretable as analogs of natural ‘coordinate forms’ in classical theory of frame bundles. We see that $\mathfrak{hor}_P^0 = \mathcal{B}$. The maps F and $\varkappa: \mathbb{V}^\wedge \rightarrow \mathbb{V}^\wedge \otimes \mathcal{A}$ naturally combine to a unital $*$ -homomorphism $F^\wedge: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P \otimes \mathcal{A}$ satisfying

$$\begin{aligned} (\text{id} \otimes \phi)F^\wedge &= (F^\wedge \otimes \text{id})F^\wedge \\ (\text{id} \otimes \epsilon)F^\wedge &= \text{id}. \end{aligned}$$

The map F^\wedge plays the role of the right action of G on horizontal forms. The corresponding F^\wedge -fixed-point graded $*$ -subalgebra $\Omega_M \subseteq \mathfrak{hor}_P$ plays the role of the differential forms on the base manifold M . Accordingly $\Omega_M^0 = \mathcal{V}$.

The map $\nabla: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ completes the picture of a frame bundle. It corresponds to the Levi-Civita connection in classical theory. By definition, ∇ intertwines the action F^\wedge and satisfies

$$\nabla\{\nabla(\mathcal{V})\mathbb{V}\} = \{0\}.$$

In particular ∇ vanishes on the subalgebra generated by $\nabla(\mathcal{V})$ and \mathbb{V}^\wedge .

Finally, it is assumed that there exist linear maps $b_\alpha: \theta \mapsto b_\alpha(\theta) \in \mathcal{B}$ and elements $f_\alpha \in \mathcal{V}$ satisfying

$$(2.5) \quad 1 \otimes \theta = \sum_\alpha b_\alpha(\theta) \nabla(f_\alpha)$$

$$(2.6) \quad F[b_\alpha(\theta)] = (b_\alpha \otimes \text{id})\varkappa(\theta).$$

This is a kind of a completeness condition. From the above mentioned postulates, it follows that ∇ is reduced in Ω_M , and that the restriction map $\overset{M}{d}: \Omega_M \rightarrow \Omega_M$ is a hermitian differential (corresponding to the standard exterior derivative of differential forms). Moreover, it can be shown that the algebra Ω_M is generated by \mathcal{V} and $\overset{M}{d}(\mathcal{V})$.

We can introduce an explicit ‘coordinate’ description of ∇ , by the formula

$$(2.7) \quad \nabla(b) = \sum_i \partial_i(b) \otimes \theta_i,$$

where $b \in \mathcal{B}$. The maps $\partial_i: \mathcal{B} \rightarrow \mathcal{B}$ are counterparts of canonical horizontal coordinate vectors fields. They completely determine ∇ . In particular, the F^\wedge -covariance of ∇ is equivalent to the property

$$(2.8) \quad F\partial_i(b) = \sum_{jk} \partial_j(b_k) \otimes c_k \kappa^{-1}(\varkappa_{ij}),$$

where $F(b) = \sum_k b_k \otimes c_k = F(b)$ and $\sum_j \theta_j \otimes \varkappa_{ji} = \varkappa(\theta_i)$. The graded Leibniz rule for ∇ is translated into the system of equations

$$(2.9) \quad \partial_i(qb) = q\partial_i(b) + \sum_j \partial_j(q)\mu_{ji}(b),$$

where $\mu: \mathcal{B} \rightarrow \text{M}_d(\mathcal{B})$ is a monomorphism defining the right \mathcal{B} -module structure on \mathfrak{hor}_P . In other words,

$$\theta_i b = \sum_j \mu_{ij}(b)\theta_j.$$

The maps $\mu_{ij}: \mathcal{B} \rightarrow \mathcal{B}$ are expressible via the right \mathcal{A} -module structure on \mathbb{V} and the map F as

$$(2.10) \quad \mu_{ij}(b) = \sum_k b_k \nu_{ij}(c_k),$$

where $\nu: \mathcal{A} \rightarrow M_d(\mathbb{C})$ is a unital homomorphism given by

$$(2.11) \quad \theta_i \circ a = \sum_j \nu_{ij}(a) \theta_j, \quad \mu(f) = f I_d \quad f \in \mathcal{V}.$$

The introduced coordinate vector fields fit into a general framework for quantum vector fields, introduced in [1].

The following identity expresses the compatibility between \circ and the $*$ -structure on \mathbb{V} :

$$(2.12) \quad \nu[\kappa(a)^*] = C_{\varkappa}^{-1/2} \overline{\nu(a)} C_{\varkappa}^{1/2}.$$

The compatibility between μ and the $*$ -structure on \mathfrak{hor}_P reads

$$(2.13) \quad \delta_{ij} b^* = \sum_k \{C_{\varkappa}^{1/2} \mu C_{\varkappa}^{-1/2}\}_{kj} \{\mu_{ik}(b)^*\}.$$

Along these lines, it is also possible to express the hermiticity property of ∇ in terms of the maps ∂_i —a direct computation gives

$$(2.14) \quad \partial_i = \sum_j [C_{\varkappa}^{1/2} \mu]_{ji} \{\partial_j^*\} \quad \partial_j^* = * \partial_i *.$$

The structure group G corresponds to a transformation group of ‘local orthonormal frames’. The frame structure allows us to think of P as the bundle of ‘orthonormal frames’ over M . In accordance with this analogy, it would be natural to assume that \varkappa is faithful (in other words G is completely determined by its action on \mathbb{V} , which means that \mathcal{A} is generated by the matrix elements of \varkappa). However, from the point of view of our spinorial constructions it is natural to allow the situations where \varkappa is not faithful ($G \leftrightarrow \text{Spin}(m)$). This allows us to include the spin structures within the framework of the frame structures.

Lemma 2.1. *In the case of ‘truly frame bundles’ when \varkappa is faithful the freeness condition for F is automatically fulfilled.*

Proof. Let us assume that \varkappa is faithful and that all the conditions on the frame structure are satisfied, except possibly the freeness of F . Let us first observe that

$$\sum_{\alpha} b_{\alpha j} F(q_{\alpha i}) = 1 \otimes \kappa^{-1}(\varkappa_{ij})$$

as it follows from (2.5) and (2.8). Here $q_{\alpha i} = \partial_i(f_{\alpha})$ and $b_{\alpha i} = b_{\alpha}(\theta_i)$.

This implies that all the elements of the form $b \otimes \varkappa_{ij}$ are in the image of the canonical map $X: \mathcal{B} \otimes_{\mathcal{V}} \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$. Now playing with the elementary [4] properties of X and the fact that \mathcal{A} is generated by \varkappa_{ij} , we conclude that X is surjective. \square

We are now going to write down very important commutation relations between coordinate vector fields ∂_i , involving the curvature tensor. Let us recall that the curvature $\varrho_{\nabla}: \mathcal{A} \rightarrow \mathfrak{h}_P^2$ of ∇ is uniquely determined through a fundamental identity

$$(2.15) \quad \nabla^2(b) = - \sum_k b_k \varrho_{\nabla}(c_k).$$

Here \mathfrak{zh}_P is the graded commutant of Ω_M in \mathfrak{hor}_P . The fact that the curvature always take values from \mathfrak{zh}_P^2 implies strong constraints for possible forms of a Levi-Civita connection in non-commutative geometry.

The canonical inclusion of the exterior algebra into the tensor algebra allows us to define the ‘components’ $\varrho_{\nabla}^{ij}: \mathcal{A} \rightarrow \mathcal{B}$ of the curvature, by the formula

$$(2.16) \quad \varrho_{\nabla}(a) = \frac{1}{2} \sum_{ij} \varrho_{\nabla}^{ij}(a) \otimes \{\theta_i \otimes \theta_j\}.$$

The components by construction satisfy a τ -antisymmetry relations

$$(2.17) \quad \varrho_{\nabla}^{ij} = - \sum_{kl} \tau_{kl}^{ij} \varrho_{\nabla}^{kl}$$

where $\tau(\theta_i \otimes \theta_j) = \sum_{kl} \tau_{ij}^{kl} \theta_k \otimes \theta_l$.

Lemma 2.2. *The following identity holds:*

$$(2.18) \quad \partial_i \partial_j(b) - \sum_{kl} \sigma_{kl}^{ij} \partial_k \partial_l(b) + \frac{1}{2} \sum_{\alpha} b_{\alpha} \varrho_{\nabla}^{ij}(c_{\alpha}) = 0,$$

where $\sum_{kl} \sigma_{ij}^{kl} \theta_k \otimes \theta_l = \sigma(\theta_i \otimes \theta_j)$.

Proof. This identity follows by rewriting (2.15) in the coordinate form, using the definitions of τ and σ , and the above coordinate expression for the curvature. \square

2.2. Integration Operators

Let us start by formulating some additional geometrical assumptions about the quantum base space M . At first, we shall assume that \mathcal{V} is realized as an everywhere dense *-subalgebra of a unital C*-algebra $\widehat{\mathcal{V}}$. We shall also assume that a faithful state $\omega_M: \widehat{\mathcal{V}} \rightarrow \mathbb{C}$ is given (representing a ‘measure’ on M). By definition, the faithfulness property means that ω_M is *strictly positive* on the positive elements. Furthermore, we shall assume that ω_M admits a modular operator $\Theta: \mathcal{V} \rightarrow \mathcal{V}$ of the form

$$(2.19) \quad \omega_M(fg) = \omega_M(\Theta(g)f) \quad \forall f, g \in \mathcal{V}.$$

The modular operator Θ is uniquely determined by the state ω_M . The following identities hold

$$\begin{aligned} \Theta(fg) &= \Theta(f)\Theta(g) \\ \Theta * \Theta &= * \end{aligned}$$

and in particular we see that Θ is necessarily bijective.

Next, we shall introduce compatibility conditions between the quantum principal bundle P with the corresponding frame structure, and the measure ω_M with the associated modular automorphism Θ .

Let us introduce a ‘vertical integration’ operator $\int_{\uparrow}: \mathcal{B} \rightarrow \mathcal{V}$ by

$$(2.20) \quad i\left[\int_{\uparrow}(b)\right] = (\text{id} \otimes h)F(b),$$

where $h: \mathcal{A} \rightarrow \mathbb{C}$ is the Haar measure [20] of G . Our first assumption on P will be:

The Strict Positivity

The map \int_{\uparrow} is strictly positive. In other words,

$$\int_{\uparrow}(b^*b) \geq 0 \quad \forall b \in \mathcal{B}$$

and if $\int_{\uparrow}(b^*b) = 0$ then $b = 0$.

This property ensures that the $*$ -algebra \mathcal{B} is closable into a C^* -algebra $\widehat{\mathcal{B}}$, using a natural GNS-type faithful $*$ -representation by bounded operators. The representation is constructed with the help of \int_{\uparrow} and ω_M , as discussed in Appendix A. By combining \int_{\uparrow} and ω_M we arrive to a faithful state $\omega_P: \widehat{\mathcal{B}} \rightarrow \mathbb{C}$ playing the role of the ‘measure’ on the bundle P . Let us assume that ω_P admits a modular operator $\widehat{\Theta}: \mathcal{B} \rightarrow \mathcal{B}$.

Lemma 2.3. *We have*

$$(2.21) \quad F\widehat{\Theta} = (\widehat{\Theta} \otimes \kappa^2)F.$$

In particular $\widehat{\Theta}[\mathcal{V}] = \mathcal{V}$ and $\widehat{\Theta} \upharpoonright \mathcal{V} = \Theta$.

Proof. The invariance property of ω_P can be written in the form

$$\sum_l \omega_P(bq_l) \otimes \kappa(d_l) = \sum_k \omega_P(b_kq) \otimes c_k.$$

This, together with the definition of $\widehat{\Theta}$ gives

$$\begin{aligned} \sum_l \omega_P[\widehat{\Theta}(q_l)b] \otimes \kappa(d_l) &= \sum_l \omega_P(bq_l) \otimes \kappa(d_l) = \sum_k \omega_P(b_kq) \otimes c_k \\ &= \sum_k \omega_P(\widehat{\Theta}(q)b_k) \otimes c_k = \omega_P(z_jb) \otimes \kappa^{-1}(e_j), \end{aligned}$$

where $F\widehat{\Theta}(q) = \sum_j z_j \otimes e_j$. Hence, it follows that $F\widehat{\Theta}(q) = \sum_l \widehat{\Theta}(q_l) \otimes \kappa^2(d_l)$. \square

The space \mathcal{B} is equipped with a natural F -invariant scalar product given by

$$\langle b, q \rangle = \omega_P(b^*q).$$

Playing with the definitions of $\widehat{\Theta}$ and \langle, \rangle it follows that $*$: $\mathcal{B} \rightarrow \mathcal{B}$ is formally adjointable. Accordingly,

$$\begin{aligned} \langle b, q^* \rangle &= \langle q, [*]^{\dagger}(b) \rangle \quad \forall q, b \in \mathcal{B} \\ [*]^{\dagger} &= \widehat{\Theta}^{-1}* = *\widehat{\Theta}. \end{aligned}$$

Let \mathcal{T} be a complete set of mutually non-equivalent irreducible representations of G . By decomposing \mathcal{B} into the multiple irreducible submodules relative to the action F , we arrive to

$$\mathcal{B} = \sum_{\alpha \in \mathcal{T}}^{\oplus} \mathcal{B}^{\alpha} \quad \mathcal{B}^{\alpha} \leftrightarrow \mathcal{E}_{\alpha} \otimes H_{\alpha}.$$

Here we have used intertwiner bimodules $\mathcal{E}_{\alpha} = \text{Mor}(\alpha, F)$, and H_{α} are the corresponding representation spaces. If \emptyset is the trivial representation in \mathbb{C} then $\mathcal{B}^{\emptyset} \leftrightarrow \mathcal{V}$. For each $\alpha \in \mathcal{T}$ let us denote by $\{\}_{\alpha}: \mathcal{B} \rightarrow \mathcal{B}^{\alpha}$ the corresponding projection map. The vertical integration map is given by projecting \mathcal{B} onto \mathcal{V} in other words $\int_{\uparrow} \leftrightarrow \{\}_{\emptyset}$. Our second extra assumption on P will be:

The Horizontal Homogeneity

Let us consider the elements $b_{\alpha i} = b_{\alpha}(\theta_i)$. Then we have

$$(2.22) \quad \omega_M \left\{ \sum_{ij} [C_{\mathcal{X}}^{-1}]_{ji} \partial_i (b_{\alpha j} f) \right\} = 0, \quad \forall f \in \mathcal{V}.$$

This condition expresses the idea that the measure on M is ‘homogeneous’. As we have seen, the maps $\partial_i: \mathcal{B} \rightarrow \mathcal{B}$ play the role of canonical horizontal vector fields, and ω_M should be invariant under the appropriate ‘infinitesimal horizontal transformations’. Let us observe that first-order ‘differential’ operators $T_{\alpha}: \mathcal{V} \rightarrow \mathcal{V}$ defined by

$$T_{\alpha}(f) = \sum_{ij} [C_{\mathcal{X}}^{-1}]_{ji} \partial_i (b_{\alpha j} f)$$

figure in the above expression. These operators are naturally associated to the frame structure.

Lemma 2.4. *Under the above homogeneity and positivity assumptions, we have*

$$(2.23) \quad \omega_P [\partial_i(b)] = 0 \quad \forall b \in \mathcal{B}.$$

Proof. Let us observe that (2.23) is non-trivial only for the F -invariant component $\{\partial_i(b)\}_{\emptyset}$ —all other components of $\partial_i(b)$ are already annihilated by the vertical integration map \int_{\cdot} . On the other hand, it is easy to see that

$$\{\partial_i(b)\}_{\emptyset} = \sum_j \partial_j(q_j),$$

where the elements q_j form an appropriate F -multiplet

$$F(q_j) = \sum_k q_k \otimes \kappa^{-2}(u_{kj}).$$

All such F -multiplets are of the form

$$q_j = \sum_{i\alpha} [C_{\mathcal{X}}^{-1}]_{ij} b_{\alpha i} f_{\alpha}$$

where $f_{\alpha} \in \mathcal{V}$ are arbitrary. Now the statement of the lemma follows from the homogeneity assumption. \square

It is interesting to calculate the formally adjoint operators for important coordinate maps $\mu_{ij}: \mathcal{B} \rightarrow \mathcal{B}$ and $\partial_i: \mathcal{B} \rightarrow \mathcal{B}$.

Lemma 2.5. *The maps μ_{ij} and ∂_i are formally adjointable and*

$$(2.24) \quad \mu^{\dagger}(b) = C_{\mathcal{X}}^{1/2} \mu(b) C_{\mathcal{X}}^{-1/2}$$

$$(2.25) \quad -\partial_i^{\dagger} = \sum_j [C_{\mathcal{X}}^{-1/2}]_{ji} \partial_j = \sum_j \mu_{ji}^{\dagger} \{\partial_j^*\}.$$

Proof. Let us first calculate the adjoint of μ_{ij} . Applying (2.10) and (2.12), and the F -invariance of the scalar product \langle, \rangle we find

$$\begin{aligned} \langle q, \mu_{ij}(b) \rangle &= \sum_k \langle q, b_k \rangle \nu_{ij}(c_k) = \sum_l \langle q_l, b \rangle \nu_{ij}[\kappa(d_l)^*] \\ &= \sum_l \langle q_l, b \rangle \left\{ C_{\mathcal{X}}^{-1/2} \overline{\nu(d_l)} C_{\mathcal{X}}^{1/2} \right\}_{ij} = \sum_l \langle q_l \left\{ C_{\mathcal{X}}^{1/2} \nu(d_l) C_{\mathcal{X}}^{-1/2} \right\}_{ij}, b \rangle \\ &= \left\{ C_{\mathcal{X}}^{1/2} \mu(q) C_{\mathcal{X}}^{-1/2} \right\}_{ij}, b \rangle \end{aligned}$$

which proves (2.24). Next, applying (2.23) and (2.9) we obtain

$$\begin{aligned} 0 &= \omega_P \partial_i(q^*b) = \omega_P \left\{ q^* \partial_i(b) \right\} + \sum_j \omega_P \left\{ \partial_j(q^*) \mu_{ji}(b) \right\} = \\ &= \langle q, \partial_i(b) \rangle + \sum_j \langle \partial_j^*(q), \mu_{ji}(b) \rangle \end{aligned}$$

and thus the second equality in (2.25) holds. The first equality in (2.25) directly follows from the second, together with (2.24) and (2.14). \square

We are going to construct the integration operator for horizontal forms. This will be done by combining the measure ω_P and a ‘coordinate volume form’ the existence of which will be ensured by our next extra condition:

Self-Duality of Coordinate Forms

There exists a number $m \in \mathbb{N}$ such that $\mathbb{V}^{\wedge m} \leftrightarrow \mathbb{C}$ and $\mathbb{V}^{\wedge k} = \{0\}$ for each $k > m$.

In other words, we can introduce the ‘volume element’ as a single generator $w = w^* \in \mathbb{V}^{\wedge m}$.

Actually, the above condition follows from a simple assumption that the braided exterior algebra \mathbb{V}^\wedge is finite-dimensional. To see this, let us consider an auxiliary braided Clifford algebra \mathcal{D} associated to $\{\mathbb{V}, \mathbb{V}^*, \sigma\}$. Here it is assumed that σ acts on all possible tensor products involving \mathbb{V} and \mathbb{V}^* . Braided exterior algebras \mathbb{V}^\wedge and $\mathbb{V}^{\wedge*} = [\mathbb{V}^*]^\wedge$ are subalgebras of \mathcal{D} and we have the following natural vector space identifications

$$\mathcal{D} \leftrightarrow \mathbb{V}^\wedge \otimes \mathbb{V}^{\wedge*} \leftrightarrow \mathbb{V}^{\wedge*} \otimes \mathbb{V}^\wedge$$

induced by the product map. The algebra \mathcal{D} naturally acts on the space \mathbb{V}^\wedge —the action is constructed with the help of the canonical duality between \mathbb{V}^\wedge and $\mathbb{V}^{\wedge*} = [\mathbb{V}^*]^\wedge$. Explicitly, we have

$$\mathbb{V}^\wedge = \mathcal{D} \otimes_{*\wedge} \mathbb{C}$$

at the level of left \mathcal{D} -modules. In a similar manner, we can construct the dual module, by considering

$$\mathbb{V}^{\wedge*} = \mathcal{D} \otimes_{\wedge} \mathbb{C}$$

and in both formulas \mathbb{C} is viewed as a trivial module over the corresponding exterior algebra. Both modules are irreducible. Now if \mathbb{V}^\wedge is finite-dimensional, then \mathcal{D} will be the full endomorphism algebra, and as such it will be allowed to have only one irreducible representation, up to equivalence. In particular, there exists a bijective \mathcal{D} -linear map $Y: \mathbb{V}^\wedge \rightarrow \mathbb{V}^{\wedge*}$.

This map is a blueprint for the Hodge $*$ -operator. It intertwines contractions and wedge-products, and in particular it follows that $Y[\mathbb{V}^{\wedge k}] = \mathbb{V}^{\wedge m-k*}$ where $m \in \mathbb{N}$ is the dimension giving the volume element. The formula

$$(2.26) \quad j(x, y)w = x \wedge y$$

defines a nondegenerate pairing $j: \mathbb{V}^{\wedge k} \times \mathbb{V}^{\wedge m-k} \rightarrow \mathbb{C}$. Moreover, it follows that there exists a unique grade-preserving map $\diamond: \mathbb{V}^\wedge \rightarrow \mathbb{V}^\wedge$ such that

$$(2.27) \quad j(y, x) = (-)^{\partial x \partial y} j(\diamond(x), y) \quad \forall x, y \in \mathbb{V}^\wedge.$$

From the definition of w it follows that

$$(2.28) \quad \varkappa(w) = w \otimes Q \quad w \circ a = \lambda(a)w,$$

where $Q \in \mathcal{A}$ is a ‘quantum determinant’ such that

$$\phi(Q) = Q \otimes Q, \quad \kappa(Q) = Q^{-1} = Q = Q^*$$

and $\lambda: \mathcal{A} \rightarrow \mathbb{C}$ is a unital multiplicative functional satisfying

$$\overline{\lambda(a)} = \lambda[\kappa(a)^*].$$

It is easy to see that the introduced objects satisfy

$$\begin{aligned} \diamond(x \circ a) &= \diamond(x) \circ \kappa^2(a) \\ \varkappa \diamond &= (\diamond \otimes \text{id}) \varkappa \\ Qa &= \text{ad}(a, \lambda)Q \\ j(x \circ a^{(1)}, y \circ a^{(2)}) &= \lambda(a)j(x, y) \\ * \diamond * &= \diamond^{-1} \quad \overline{j(x, y)} = (-)^{\partial x \partial y} j(y^*, x^*). \end{aligned}$$

Let us observe that in general $m \neq d$, in contrast with classical geometry.

Now we shall introduce the integration map $\int_P: \mathfrak{hor}_P \rightarrow \mathbb{C}$. The definition is straightforward

$$(2.29) \quad \int_P (b \otimes \vartheta) = \begin{cases} \omega_P(b) & \text{for } \vartheta = w, \\ 0 & \text{if } \deg \vartheta < m. \end{cases}$$

Lemma 2.6. *We have*

$$(2.30) \quad \int_P \nabla(\varphi) = 0 \quad \forall \varphi \in \mathfrak{hor}_P.$$

Moreover,

$$(2.31) \quad \int_P [\varphi^*] = \left(\int_P \varphi \right)^* \quad \sum_k \left(\int_P \varphi_k \right) \otimes c_k = \left(\int_P \varphi \right) \otimes Q,$$

where $\sum_k \varphi_k \otimes c_k = F^\wedge(\varphi)$.

Proof. Property (2.30) follows directly from (2.23) and the definition of \int_P . Hermiticity and covariance of \int_P are simple consequences of the similar properties for the measure ω_P . \square

Let us now observe that Q , as a hermitian involution, naturally decomposes into hermitian projections Q_+, Q_- where $Q_\pm = 1/2 \pm Q/2$. If the map $\lambda: \mathcal{A} \rightarrow \mathbb{C}$ is in addition *central*, then it will be possible to pass to the corresponding ‘components’ of G , determined by Q_\pm . In particular, we can factorize through the Hopf $*$ -ideal generated by Q_- , reducing to the case $Q = 1$. This is the quantum version of *unimodularity*—passing from $O(d)$ to $SO(d)$ groups.

Geometrically, such a restriction means that we are dealing with *oriented manifolds*. In what follows it will be assumed that orientability property holds (and centrality of λ , as a necessary consistency condition).

For the end of this subsection, we shall analyze modular properties of the integration map.

Lemma 2.7. *The following identity holds*

$$(2.32) \quad \int_P \varphi \psi = (-)^{\partial \varphi \partial \psi} \int_P \Lambda(\psi) \varphi$$

where $\Lambda: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ is a grade-preserving homomorphism defined by $\Lambda(b \otimes \vartheta) = \widehat{\Theta}(b_\lambda) \otimes \diamond(\vartheta)$ and $\mathbb{I}_\lambda = (\text{id} \otimes \lambda)F$.

Proof. Assuming that $\varphi = b \otimes \vartheta$ and $\psi = q \otimes \eta$ and performing elementary transformations we find

$$\begin{aligned}
\int_P \varphi \psi &= \sum_l \int_P \{b q_l \otimes (\vartheta \circ d_l) \eta\} = \sum_l \omega_P(b q_l) j(\vartheta \circ d_l, \eta) \\
&= \sum_k \omega_P(\widehat{\Theta}(q) b_k) j(\vartheta \circ \kappa^{-1}(c_k), \eta) = \sum_k (-)^{\partial \varphi \partial \psi} \omega_P(\widehat{\Theta}(q) b_k) j(\diamond(\eta), \vartheta \circ \kappa^{-1}(c_k)) \\
&= \sum_k (-)^{\partial \varphi \partial \psi} \omega_P(\widehat{\Theta}(q) b_k) j(\diamond(\eta) \circ c_k^{(1)} \lambda^{-1}(c_k^{(2)}), \vartheta) \\
&= \sum_k (-)^{\partial \varphi \partial \psi} \omega_P(\widehat{\Theta}(q_\lambda) b_k) j(\diamond(\eta) \circ c_k, \vartheta) = (-)^{\partial \varphi \partial \psi} \int_P \{ \Lambda(\psi) \varphi \}.
\end{aligned}$$

Let us also observe that

$$F^\wedge \Lambda = (\Lambda \otimes \kappa^2) F^\wedge \quad * \Lambda * = \Lambda^{-1}.$$

The map Λ is uniquely determined by (2.32). \square

Through this paper we shall make an extensive use of an *extended horizontal forms* algebra $\mathfrak{hor}_{P,\Sigma}$ obtained by mixing Σ with the standard horizontal forms. More precisely, $\mathfrak{hor}_{P,\Sigma}$ is obtained by taking the twisted tensor product between \mathcal{B} and the extended braided exterior algebra \mathbb{V}_Σ^\wedge . We have obviously natural left/right \mathcal{B}, Σ -module identifications

$$\mathfrak{hor}_{P,\Sigma} \leftrightarrow \Sigma \otimes \mathfrak{hor}_P \leftrightarrow \mathfrak{hor}_P \otimes \Sigma.$$

The action F^\wedge naturally extends, with the help of $\kappa_\Sigma: \Sigma \rightarrow \Sigma \otimes \mathcal{A}$, to the action $F^\wedge: \mathfrak{hor}_{P,\Sigma} \rightarrow \mathfrak{hor}_{P,\Sigma} \otimes \mathcal{A}$. The frame structure ∇ naturally extends, by Σ -left/right linearity, to a hermitian antiderivation $\nabla: \mathfrak{hor}_{P,\Sigma} \rightarrow \mathfrak{hor}_{P,\Sigma}$.

The base space algebra Ω_M is naturally included in $\Omega_{M,\Sigma}$ —which is defined as the F^\wedge -fixed point subalgebra of $\mathfrak{hor}_{P,\Sigma}$. We shall freely pass from extended to non-extended objects, and vice versa.

2.3. The Hodge Operator—Linear Algebra

We shall first introduce the Hodge $*$ -operator on \mathbb{V}_Σ^\wedge . Then it will be extended to the level of $\mathfrak{hor}_{P,\Sigma}$. We shall assume that \mathbb{V}_Σ^\wedge is equipped with a Σ -valued quadratic form g_\wedge and the associated scalar product $\langle \rangle$, as explained in Appendix B. We have

$$\mathbb{V}_\Sigma^{\wedge m} = \Sigma w = w \Sigma \quad w \alpha = S(\alpha) w$$

where $S: \Sigma \rightarrow \Sigma$ is an automorphism satisfying

$$\begin{aligned}
S(\alpha) \circ a &= S(\alpha \circ a) \quad * S * = S^{-1} \\
\kappa_\Sigma S &= (S \otimes \text{id}) \kappa_\Sigma.
\end{aligned}$$

The map j is straightforwardly extendible to a Σ -valued pairing acting within \mathbb{V}_Σ^\wedge . We have

$$\begin{aligned}
j(\varphi, \psi q) &= j(\varphi, \psi) S(q) \quad j(q \varphi, \psi) = q j(\varphi, \psi) \\
j(\varphi, q \psi) &= j(\varphi q, \psi) \\
j(\varphi \circ a^{(1)}, \psi \circ a^{(2)}) &= \lambda(a^{(1)}) j(\varphi, \psi) \circ a^{(2)}
\end{aligned}$$

Proposition 2.8. (i) *The formula*

$$(2.33) \quad g_{\wedge}(x, y) = j(x, \star[y])$$

uniquely defines a linear operator $\star: \mathbb{V}_{\Sigma}^{\wedge} \rightarrow \mathbb{V}_{\Sigma}^{\wedge}$ such that $\star(\mathbb{V}_{\Sigma}^{\wedge k}) \subseteq \mathbb{V}_{\Sigma}^{\wedge m-k}$.

(ii) *The map \star is bijective, and we have*

$$(2.34) \quad \star[xq] = \star[x]S^{-1}(q) \quad \star[qx] = q\star[x] \quad q \in \Sigma$$

$$(2.35) \quad \varkappa\star = (\star \otimes \text{id})\varkappa$$

$$(2.36) \quad \star(\vartheta \circ a_{\lambda}) = \star(\vartheta) \circ a.$$

Proof. The \varkappa -covariance of \star follows from formula (2.33) and the covariance of all defining entities. The Σ -compatibility follows from a similar property for j . Finally, performing elementary transformations we obtain

$$\begin{aligned} g_{\wedge}(\xi, \vartheta) \circ a &= g_{\wedge}(\xi \circ a^{(1)}, \vartheta \circ a^{(2)}) = j(\xi \circ a^{(1)}, \star[\vartheta \circ a^{(2)}]) = j(\xi, \star[\vartheta]) \circ a \\ &= j(\xi \circ a^{(1)}, \star[\vartheta] \circ a^{(2)}) \lambda^{-1}(a^{(3)}), \end{aligned}$$

and hence (2.36) holds. \square

The automorphism $S: \Sigma \rightarrow \Sigma$ has a simple structure, as it is sufficient to calculate its action on the elements of the form $g(x, y)$ where $x, y \in \mathbb{V}$.

By construction, we have $\sigma: w \otimes x \mapsto T(x) \otimes w$ and $\sigma: y \otimes w \mapsto w \otimes T^*(y)$ where $T: \mathbb{V} \rightarrow \mathbb{V}$ is a bijective linear operator. Using this, and iteratively applying the definition of the Σ -bimodule structure on \mathbb{V}_{Σ} we find

$$(2.37) \quad S\{\langle x, y \rangle\} = \langle T^{-1}(x), T(y) \rangle.$$

The map T naturally extends to a unital automorphism $T: \mathbb{V}^{\wedge} \rightarrow \mathbb{V}^{\wedge}$, and furthermore to $T: \mathbb{V}_{\Sigma}^{\wedge} \rightarrow \mathbb{V}_{\Sigma}^{\wedge}$ by imposing $T(xq) = T(x)S(q)$ and $T(qy) = S(q)T(y)$. The above formula remains valid for arbitrary elements of our braided exterior algebra.

Lemma 2.9. *We have*

$$(2.38) \quad \star^{\dagger} = \star, \quad T^{\dagger} = T.$$

In other words,

$$\langle x, T(y) \rangle = S\{\langle T(x), y \rangle\} \quad \langle x, \star(y) \rangle = S^{-1}\{\langle \star(x), y \rangle\}$$

Here, the concept of the adjoint operator is appropriately S -twisted—a necessary consistency condition, having in mind a right S -twisted Σ -linearity of \star and T .

Proof. In fact, equation (2.37) expresses the selfadjointness of T . We compute

$$[x^*y]_m^* = w\langle x, \star^{-1}(y) \rangle^* = S\{\langle \star^{-1}(y), x \rangle\}w = [y^*x]_m = \langle y, \star^{-1}(x) \rangle w.$$

This shows that \star is selfadjoint. \square

We conclude this subsection by connecting the contraction and multiplication operators in $\mathbb{V}_{\Sigma}^{\wedge}$. By definition, contraction operators $\iota[x]: \mathbb{V}_{\Sigma}^{\wedge} \rightarrow \mathbb{V}_{\Sigma}^{\wedge}$ are given by

$$(2.39) \quad \iota[x]\psi = (g \otimes \text{id}^{n-1})(x \otimes \psi) \quad \forall \psi \in \mathbb{V}_{\Sigma}^{\wedge n},$$

where ψ are realized in the tensor algebra.

It is easy to see that these operators are σ -braided antiderivations (they satisfy the σ -braided Leibniz rule). In other words,

$$(2.40) \quad \iota[x]y + \sum_{\alpha} y_{\alpha} \iota[x_{\alpha}] = g(x, y) \quad \forall x, y \in \mathbb{V}_{\Sigma}$$

where $\sum_{\alpha} y_{\alpha} \otimes x_{\alpha} = \sigma(x \otimes y)$.

Furthermore, using a natural Σ -valued scalar product in $\mathbb{V}_{\Sigma}^{\wedge}$ it follows that

$$(2.41) \quad [x \wedge ()]^{\dagger} = \iota[x^*] \quad \forall x \in \mathbb{V}_{\Sigma}.$$

In other words, the contraction operators are the adjoint maps of the corresponding multiplication operators.

Lemma 2.10. *The following identity holds*

$$(2.42) \quad \iota[e] = \star^{-1}[e \wedge ()]\star.$$

Proof. We have

$$[x^* \star \iota[e](y)]_m = \langle x, \iota[e](y) \rangle w = \langle e^* \wedge x, y \rangle w = [x^* \{e \wedge \star[y]\}]_m$$

and thus (2.42) follows. \square

In other words \star acts as a conjugation between multiplication and contraction maps.

2.4. Extension To Horizontal Forms

The operator \star will be extended to $\mathfrak{hor}_{P,\Sigma}$ by left \mathcal{B} -linearity, in other words we define

$$(2.43) \quad \star_P(b \otimes \vartheta) = b \otimes \star(\vartheta) \quad \forall b \in \mathcal{B} \quad \forall \vartheta \in \mathbb{V}_{\Sigma}^{\wedge}.$$

Of course, here it is necessary to deal with extended horizontal forms $\mathfrak{hor}_{P,\Sigma}$. As a consequence of (2.35) we have

$$(2.44) \quad F^{\wedge} \star_P = (\star_P \otimes \text{id}) F^{\wedge}.$$

This intertwining property, together with (2.36) and the definition of the product in $\mathfrak{hor}_{P,\Sigma}$, implies that \star_P is λ -twisted right \mathcal{B} -linear, $\star_P(\psi b_{\lambda}) = \star_P(\psi)b$.

Definition 1. The map \star_P is called the *Hodge *-operator* for P .

Let us observe that $\star_P(\Omega_{M,\Sigma}) = \Omega_{M,\Sigma}$, as directly follows from (2.44). We shall denote by $\star_M: \Omega_{M,\Sigma} \rightarrow \Omega_{M,\Sigma}$ the corresponding restricted map.

The introduced integration map $\int_P: \mathfrak{hor}_P \rightarrow \mathbb{C}$ naturally extends, by left Σ -linearity, to $\int_P: \mathfrak{hor}_{P,\Sigma} \rightarrow \Sigma$. Such an extended map intertwines the actions of G and satisfies

$$\int_P[\varphi^*] = S\left\{\int_P[\varphi]^*\right\}, \quad \int_P[\varphi q] = \int_P[\varphi]S(q) \quad q \in \Sigma.$$

The Hodge *-operator, together with the extended integration map, enables us to introduce naturally a scalar product in the algebra of horizontal forms.

Lemma 2.11. *The formula*

$$(2.45) \quad \langle \varphi, \psi \rangle = \int_P \varphi^* \star_P[\psi]$$

defines a Σ -valued scalar product in $\mathfrak{hor}_{P,\Sigma}$. This scalar product is G -covariant, and in terms of the natural left \mathcal{B} -module identification $\mathfrak{hor}_{P,\Sigma} \leftrightarrow \mathcal{B} \otimes \mathbb{V}_{\Sigma}^{\wedge}$ it is given by a direct product of natural scalar products in $\mathbb{V}_{\Sigma}^{\wedge}$ and \mathcal{B} .

Proof. The F^\wedge -covariance of the introduced scalar product follows from (2.44) and the F^\wedge -covariance of \int_P . Let us further observe that the defined scalar product possesses all appropriate Σ -anti/linearity properties. This follows from left Σ -linearity of \int_P and \star and identities describing S -twisted right Σ -linearity of these maps.

It is sufficient to verify the lemma on the non-correlated elements of the extended horizontal forms algebra. A direct computation gives

$$\begin{aligned} \langle b \otimes \vartheta, q \otimes \eta \rangle &= \int_P \left\{ (\vartheta^* \otimes b^*)(q \otimes \star[\eta]) \right\} = \sum_{kl} \int_P \left\{ b_k^* q_l \otimes \vartheta^* \circ (c_k^* d_l) \star[\eta] \right\} = \\ &= \sum_{kl} \omega_P[b_k^* q_l] g_\wedge(\vartheta^* \circ (c_k^* d_l), \eta) = \omega_P[b^* q] \langle \vartheta, \eta \rangle \end{aligned}$$

where $b, q \in \mathcal{B}$ and $\vartheta, \eta \in \mathbb{V}_\Sigma^\wedge$. \square

In what follows we shall assume that $\mathfrak{hor}_{P,\Sigma}$ is equipped with the constructed scalar product. The next lemma gives an explicit description of the (formal) adjoint covariant derivative map.

Lemma 2.12. *The map $\nabla: \mathfrak{hor}_{P,\Sigma} \rightarrow \mathfrak{hor}_{P,\Sigma}$ is adjointable, in other words there exists a (necessarily unique) linear map $\nabla^\dagger: \mathfrak{hor}_{P,\Sigma} \rightarrow \mathfrak{hor}_{P,\Sigma}$ such that*

$$(2.46) \quad \langle \varphi, \nabla(\psi) \rangle = \langle \nabla^\dagger(\varphi), \psi \rangle \quad \forall \varphi, \psi \in \mathfrak{hor}_P.$$

Explicitly,

$$(2.47) \quad \nabla^\dagger(\psi) = -\star_P^{-1} \nabla \star_P(\psi).$$

The map ∇^\dagger is Σ -bilinear. It intertwines the right action F^\wedge —in other words

$$(2.48) \quad F^\wedge \nabla^\dagger = (\nabla^\dagger \otimes \text{id}) F^\wedge.$$

Proof. Let us start from the identity $\int_P \nabla = 0$. We compute

$$\begin{aligned} 0 &= \int_P \nabla(\varphi^* \star_P(\psi)) \sim \int_P \left\{ \nabla(\varphi)^* \star_P(\psi) \right\} + \int_P \left\{ \varphi^* \nabla \star_P(\psi) \right\} = \\ &= \langle \nabla(\varphi), \psi \rangle + \langle \varphi, \star_P^{-1} \nabla \star_P(\psi) \rangle \end{aligned}$$

and hence ∇ is adjointable in \mathfrak{hor}_P and (2.46) holds. The F^\wedge -covariance property follows from the F^\wedge -covariance of ∇ and the F^\wedge -invariance of the scalar product. The Σ -linearity is a direct consequence of (2.47) and the Σ -linearity of ∇ . It also follows from (2.47) and the twisted Σ -linearity of \star . \square

There is a simple coordinate expression for the adjoint derivative. It is given by

$$(2.49) \quad \nabla^\dagger = - \sum_{i=1}^d \partial_i \otimes \iota[\theta_i].$$

This can be proved in several ways, for example it follows by explicitly taking the adjoints of ∂_i and θ_i in the coordinate expression for ∇ . Let us observe that ∇^\dagger generically takes the values from the extended horizontal algebra $\mathfrak{hor}_{P,\Sigma}$.

2.5. Quantum Laplacian

We are now ready to introduce a quantum Laplace operator, in a similar manner as in classical geometry.

Definition 2. A linear operator $\Delta_P: \mathfrak{hor}_{P,\Sigma} \rightarrow \mathfrak{hor}_{P,\Sigma}$ defined by

$$(2.50) \quad \Delta_P = \nabla \nabla^\dagger + \nabla^\dagger \nabla$$

is called the *quantum Laplacian*.

By construction, Δ_P is a symmetric positive operator. Note that it operates within the extended horizontal forms algebra. It is interesting to write down an explicit coordinate formula for the quantum Laplacian.

Proposition 2.13. *We have*

$$(2.51) \quad \Delta_P(b \otimes \vartheta) = - \sum_{ij} \partial_i \partial_j (b) \otimes g_{ij} \vartheta + \frac{1}{2} \sum_{ij\alpha} b_\alpha \varrho_{\nabla}^{ij}(c_\alpha) \otimes \theta_i \iota[\theta_j](\vartheta),$$

where $F(b) = \sum_\alpha b_\alpha \otimes c_\alpha$ and $g_{ij} = g(\theta_i, \theta_j)$.

Proof. Playing with coordinate formulas (2.7) and (2.49), and the commutation relations (2.18) and (2.40) we find

$$\begin{aligned} \Delta_P(b \otimes \vartheta) &= - \sum_{ij} \left\{ \partial_i \partial_j (b) \otimes \iota[\theta_i](\theta_j \vartheta) \right\} - \sum_{ij} \left\{ \partial_i \partial_j (b) \otimes \theta_i \iota[\theta_j](\vartheta) \right\} \\ &= - \sum_{ij} \partial_i \partial_j (b) \otimes g_{ij} \vartheta - \sum_{ij} \left\{ \partial_i \partial_j (b) \otimes \theta_i \iota[\theta_j](\vartheta) \right\} \\ &\quad + \sum_{ijkl} \partial_k \partial_l (b) \otimes \sigma_{kl}^{ij} \theta_i \iota[\theta_j](\vartheta) \\ &= - \sum_{ij} \partial_i \partial_j (b) \otimes g_{ij} \vartheta + \frac{1}{2} \sum_{ij\alpha} b_\alpha \varrho_{\nabla}^{ij}(c_\alpha) \otimes \theta_i \iota[\theta_j](\vartheta). \quad \square \end{aligned}$$

In general, maps $g_{ij}: \mathbb{V}_\Sigma^\wedge \rightarrow \mathbb{V}_\Sigma^\wedge$ will be non-scalar operators. Here are some further elementary algebraic properties of Δ_P —the covariance property

$$(2.52) \quad F^\wedge \Delta_P = (\Delta_P \otimes \text{id}) F^\wedge,$$

and it is worth mentioning the following expressions

$$(2.53) \quad -\Delta_P \star_P^{-1} = \nabla \star_P \nabla + \star_P^{-1} \nabla \star_P \nabla \star_P^{-1}$$

$$(2.54) \quad -\star_P \Delta_P = \nabla \star_P^{-1} \nabla + \star_P \nabla \star_P^{-1} \nabla \star_P.$$

According to (2.52) the map Δ_P is reduced in Ω_M . We shall denote the corresponding restriction map by $\Delta_M: \Omega_M \rightarrow \Omega_M$. Obviously,

$$\begin{aligned} \Delta_M {}^M d &= {}^M d \Delta_M & \Delta_M {}^M d^\dagger &= {}^M d^\dagger \Delta_M \\ \Delta_M &= ({}^M d + {}^M d^\dagger)^2. \end{aligned}$$

Let us consider a natural decomposition

$$\mathfrak{hor}_P = \sum_{\alpha \in \mathcal{T}}^\oplus \mathcal{H}^\alpha \quad \mathcal{H}^\alpha \leftrightarrow \mathcal{F}_\alpha \otimes H_\alpha$$

into the multiple irreducible subspaces. It is worth mentioning that these subspaces are mutually orthogonal with respect to the scalar product \langle, \rangle in \mathfrak{hor}_P . All the maps ∇, ∇^\dagger and Δ_P are reduced in the spaces \mathcal{H}^α .

3. QUANTUM SPIN BUNDLES

Throughout this section we shall assume that the structure group G possesses a very special ‘spinorial’ representation (and consequently we shall relax from the faithfulness assumption for \varkappa). The corresponding frame structures on quantum spaces/bundles are then interpretable as ‘covering bundles’ of the ‘real’ orthonormal frame bundles. The orthogonal quantum group G_0 corresponds to the Hopf $*$ -subalgebra \mathcal{A}_0 of \mathcal{A} generated by the matrix elements \varkappa_{ij} .

The original orthonormal frame bundle P_0 is given by the $*$ -subalgebra \mathcal{B}_0 of \mathcal{B} generated by multiple irreducible submodules of \mathcal{B} corresponding to the representations of G_0 . Obviously $F(\mathcal{B}_0) \subseteq \mathcal{B}_0 \otimes \mathcal{A}_0$ and $i(\mathcal{V}) \subseteq \mathcal{B}_0$. Taking the corresponding restriction maps, we obtain a quantum principal G_0 -bundle $P_0 = (\mathcal{B}_0, i, F)$ over M with the faithful action \varkappa . Geometrically, P is a kind of a covering space for P_0 and P_0 corresponds to the ‘vanilla’ orthonormal frame bundle in classical geometry.

We shall first formalize the idea of a ‘quantum spinor space’. We shall use the quantum Clifford algebra $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ associated to \mathbb{V} , metric coefficients algebra Σ , the braid operator σ and quantum metric $g: \mathbb{V} \otimes \mathbb{V} \rightarrow \Sigma$. This algebra is constructed by g -deforming the product in \mathbb{V}_Σ^\wedge , while preserving the $*$ -structure and the \circ -structure.

Let us assume that a finite-dimensional Hilbert space \mathbb{S} is given, together with a unitary representation $\varkappa_\mathbb{S}: \mathbb{S} \rightarrow \mathbb{S} \otimes \mathcal{A}$. Let us also assume that \mathbb{S} is an irreducible left $*$ -module over $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$. Finally, let us assume that the following compatibility condition holds:

$$(3.1) \quad \varkappa_\mathbb{S}(Z\xi) = \varkappa_\Sigma(Z)\varkappa_\mathbb{S}(\xi) \quad Z \in \text{cl}[\mathbb{V}, g, \sigma, \Sigma] \quad \xi \in \mathbb{S}.$$

The meaning of this condition is that the action of G on $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ can be viewed as the adjoint action of $\varkappa_\mathbb{S}$, in terms of operators acting in \mathbb{S} .

Definition 3. If the above conditions are fulfilled, we shall say that \mathbb{S} is a *quantum spinor space* associated to G and $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$.

The $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ -module structure $\gamma: \text{cl}[\mathbb{V}, g, \sigma, \Sigma] \rightarrow B(\mathbb{S})$ is generally not faithful, and Σ may be infinite-dimensional. The map γ (including its values on Σ) is completely determined by the assignment $\gamma: \mathbb{V} \rightarrow B(\mathbb{S})$. This simple observation can be used as a starting point in constructing Σ and $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$.

By combining the $*$ -algebra structures on \mathcal{B} and $\mathbb{V}_\Sigma^\wedge \hookrightarrow \text{cl}[\mathbb{V}, g, \sigma, \Sigma]$, we obtain a $*$ -algebra $\text{cl}[P]$. By construction, we have a natural identification $\text{cl}[P] \leftrightarrow \text{hor}_{P, \Sigma}$ of \mathcal{B} -bimodules. The $*$ -algebra structure on $\text{cl}[P]$ is given by the standard cross-product type formulae

$$\begin{aligned} (q \otimes \vartheta)(b \otimes \eta) &= \sum_k q b_k \otimes (\vartheta \circ c_k) \eta \\ (b \otimes \vartheta)^* &= \sum_k b_k^* \otimes (\vartheta^* \circ c_k^*). \end{aligned}$$

By taking the product of the actions \varkappa_* and F we obtain a $*$ -homomorphism $F_{\text{cliff}}: \text{cl}[P] \rightarrow \text{cl}[P] \otimes \mathcal{A}$. Obviously $F_{\text{cliff}} \leftrightarrow F^\wedge$, in terms of the identification

$$\text{cl}[P] \leftrightarrow \text{hor}_{P, \Sigma}.$$

We shall denote by $\text{cl}[M] \subseteq \text{cl}[P]$ the F_{cliff} -invariant $*$ -subalgebra of $\text{cl}[P]$. Obviously, we have a natural identification

$$\text{cl}[M] \leftrightarrow \Omega_{M, \Sigma}.$$

Starting from a quantum spinor space, we can define the associated spinor bundle. Let us consider a free left \mathcal{B} -bimodule \mathcal{S} , given by

$$(3.2) \quad \mathcal{S} = \mathcal{B} \otimes \mathbb{S}.$$

By taking the product of actions F and $\kappa_{\mathbb{S}}$ we obtain the map $F_{\mathcal{S}}: \mathcal{S} \rightarrow \mathcal{S} \otimes \mathcal{A}$. There exists a natural $F_{\mathcal{S}}$ -invariant scalar product on \mathcal{S} , defined by taking the direct product of the scalar products in \mathbb{S} and \mathcal{B} . In what follows we shall assume that \mathcal{S} is equipped with this scalar product. Let $\mathcal{S}_M \subseteq \mathcal{S}$ be the subspace of $F_{\mathcal{S}}$ -invariant elements. This space is a \mathcal{V} -bimodule, in a natural way. In accordance with our general discussion, it is interpretable as the appropriate associated spinor bundle.

Definition 4. The $*$ -algebra $\mathfrak{cl}[M]$ is called *quantum Clifford bundle algebra* over the space M . The \mathcal{V} -bimodule \mathcal{S}_M is called *quantum spinor bundle*, and its elements are called *quantum spinor fields* over M .

It is possible to introduce a natural action map $\beta: \mathfrak{cl}[P] \otimes \mathcal{S} \rightarrow \mathcal{S}$ of $\mathfrak{cl}[P]$ on \mathcal{S} . This map is defined by

$$(3.3) \quad (q \otimes x)(b \otimes \zeta) = \sum_k q b_k \otimes (x \circ c_k)[\zeta]$$

It is easy to see that this indeed defines a faithful unital action of $\mathfrak{cl}[P]$ on \mathcal{S} , intertwining the corresponding natural coactions $F_{\text{cliff}} \times F_{\mathcal{S}}$ and $F_{\mathcal{S}}$. In particular, it follows that

$$(3.4) \quad \beta(\mathfrak{cl}[M] \otimes \mathcal{S}_M) = \mathcal{S}_M.$$

Proposition 3.1. (i) *The action $\beta: \mathfrak{cl}[P] \otimes \mathcal{S} \rightarrow \mathcal{S}$ is hermitian, in other words*

$$(3.5) \quad \langle \psi, T\varphi \rangle = \langle T^*\psi, \varphi \rangle$$

for each $\psi, \varphi \in \mathcal{S}$ and $T \in \mathfrak{cl}[P]$.

(ii) *The operators $T: \mathcal{S} \rightarrow \mathcal{S}$ coming from $\mathfrak{cl}[P]$ are generally unbounded. However the restrictions on \mathcal{S}_M are bounded. In particular $\mathfrak{cl}[M]$ acts on \mathcal{S}_M by bounded operators.*

Proof. We compute

$$\begin{aligned} \langle b \otimes \zeta, T(q \otimes \xi) \rangle &= \sum_{\alpha} \langle b \otimes \zeta, u q_{\alpha} \otimes [\vartheta \circ d_{\alpha}](\xi) \rangle \\ &= \sum_{\alpha} \omega_P(b^* u q_{\alpha})(\zeta, [\vartheta \circ d_{\alpha}](\xi)) = \sum_{\alpha} \omega_P(b^* u q_{\alpha}) \langle [\vartheta \circ d_{\alpha}]^*(\zeta), \xi \rangle \\ &= \sum_{\alpha} \omega_P((u^* b)^* q_{\alpha}) \langle [\vartheta^* \circ \kappa(d_{\alpha})^*](\zeta), \xi \rangle \\ &= \sum_{ik} \omega_P((u_i^* b_k)^* q) \langle [\vartheta^* \circ (c_k^* a_i)^*](\zeta), \xi \rangle = \langle T^*(b \otimes \zeta), q \otimes \xi \rangle, \end{aligned}$$

where $T = u \otimes \vartheta$ and the corresponding sums indicate the action of F on the elements q, b and u . Consequently (3.5) holds. The fact that the operators T are unbounded in general, comes from the \circ -structure used in the definition of the action of $\mathfrak{cl}[P]$. Indeed, the \circ is generally *not continuous* when viewed as a homomorphism $\circ: \mathcal{A} \rightarrow M_d(\mathbb{C})$.

On the other hand, multiplication operators by $q \in \mathcal{B}$ are continuous. If $\psi = \sum_{b\zeta} b \otimes \zeta \in \mathcal{S}_M$ then it is easy to see that $(q \otimes x)[\psi] = \sum_{b\zeta} q b \otimes T_x[\zeta]$ which implies that the restricted action on \mathcal{S}_M is continuous. \square

4. QUANTUM DIRAC OPERATOR

Let us consider a linear operator $\mathbb{D}: \mathcal{S} \rightarrow \mathcal{S}$ given by

$$(4.1) \quad \mathbb{D}(b \otimes x) = -i \sum_j \partial_j(b) \otimes \theta_j[x],$$

where we have interpreted the elements of \mathbb{V} as linear operators in \mathbb{S} , in accordance with our definition of quantum Clifford algebras. We begin by demonstrating a couple of elementary properties of the introduced map.

Proposition 4.1. (i) *The map \mathbb{D} is $F_{\mathcal{S}}$ -covariant. In other words,*

$$(4.2) \quad F_{\mathcal{S}} \mathbb{D} = (\mathbb{D} \otimes \text{id}) F_{\mathcal{S}}.$$

In particular $\mathbb{D}(\mathcal{S}_M) \subseteq \mathcal{S}_M$.

(ii) *We have*

$$(4.3) \quad \langle \psi, \mathbb{D}(\varphi) \rangle = \langle \mathbb{D}(\psi), \varphi \rangle$$

for each $\psi, \varphi \in \mathcal{S}$.

Proof. The intertwining property of \mathbb{D} is an immediate consequence of its definition and the transformation properties of θ_i and ∂_i . To check (4.3) let us observe that the scalar product in \mathcal{S} is obtained by tensoring the scalar products in \mathcal{B} and \mathbb{S} . Consequently, taking the formal adjoints, applying (2.25) and playing with the basis θ_i we find

$$\begin{aligned} \mathbb{D}^\dagger &= i \sum_j (\partial_j \otimes \theta_j)^\dagger = i \sum_j \partial_j^\dagger \otimes \theta_j^\dagger = \\ &= -i \sum_{jk} [C_{\mathcal{Z}}^{-1/2}]_{kj} \partial_k \otimes C_{\mathcal{Z}}^{-1/2}(\theta_j) = -i \sum_k \partial_k \otimes \theta_k = \mathbb{D}. \quad \square \end{aligned}$$

Definition 5. The map $\mathbb{D}: \mathcal{S}_M \rightarrow \mathcal{S}_M$ is called a *quantum Dirac operator* for M .

The following simple proposition shows us that the Dirac operator contains the whole information about the differential $\overset{M}{d}: \Omega_M \rightarrow \Omega_M$, as in the classical theory. This fits into the axiomatic framework of [2]. However, in contrast to [2] the eigenvalues of our Dirac operator do not obey the classical-type asymptotics in general.

Proposition 4.2. *We have*

$$(4.4) \quad [\mathbb{D}, f] = \overset{M}{d}(f) \quad \forall f \in \mathcal{V}.$$

Here we have interpreted f and $\overset{M}{d}(f)$ as sections of the Clifford algebra bundle.

Proof. A direct computation gives

$$\begin{aligned} \mathbb{D}[f(b \otimes \vartheta)] &= \sum_i \partial_i(fb) \otimes \theta_i \vartheta = \left\{ \sum_i \partial_i(f) \otimes \theta_i \right\} (b \otimes \vartheta) \\ &\quad + f \sum_i \partial_i(b) \otimes \theta_i \vartheta = \overset{M}{d}(f)(b \otimes \vartheta) + f \mathbb{D}(b \otimes \vartheta). \quad \square \end{aligned}$$

Let us now investigate the relation between \mathbb{D} and the spinorial Laplacian.

Definition 6. In accordance with classical theory, the operator $\Delta_S: \mathcal{S}_M \rightarrow \mathcal{S}_M$ given by

$$(4.5) \quad \Delta_S = - \sum_{ij} \partial_i \partial_j \otimes g_{ij}$$

is called the *spinorial Laplacian*. Here $g_{ij} = g(\theta_i, \theta_j): \mathbb{S} \rightarrow \mathbb{S}$.

Here is a quantum version of classical Lichnerowicz formula.

Proposition 4.3. *We have*

$$(4.6) \quad \mathbb{D}^2 = \Delta_S + \hat{R}$$

where $\hat{R}: \mathcal{S}_M \rightarrow \mathcal{S}_M$ is the ‘cliffordization’ of the curvature. Explicitly $\hat{R} \leftrightarrow \nabla^2$ in terms of the identification $\text{cl}[\mathbb{V}, g, \sigma, \Sigma] \leftrightarrow \mathbb{V}_\Sigma^\wedge$.

Proof. This is a direct consequence of the definition of the quantum Dirac operator and the definition of the product in braided Clifford algebras. \square

5. EXAMPLES & REMARKS

5.1. General Connections

So far we have only considered the torsionless Levi-Civita connections. Our results are easily applicable to more general connections. By definition, a *bundle derivative* on a framed quantum principal bundle P is an arbitrary first-order hermitian antiderivation $D: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ extending the differential $\overset{M}{d}: \Omega_M \rightarrow \Omega_M$ and intertwining the action map F^\wedge . The torsion tensor $T_D: \mathbb{V} \rightarrow \mathfrak{zh}_P^2$ of a bundle derivative D is defined by

$$T_D(\theta) = D(\theta),$$

as in classical geometry. The curvature $\varrho_D: \mathcal{A} \rightarrow \mathfrak{zh}_P^2$ of D is given by

$$D^2(\varphi) = -(-)^{\partial\varphi} \sum_k \varphi_k \varrho_D(c_k).$$

The bundle derivatives form a real affine space $\mathfrak{der}(P)$.

Lemma 5.1. *We have*

$$(5.1) \quad \int_P D(\varphi) = 0 \quad \forall D \in \mathfrak{der}(P).$$

Proof. According to the results of [5] we can write

$$D(\varphi) = \nabla(\varphi) + (-)^{\partial\varphi} \sum_k \varphi_k \lambda(c_k),$$

where $\lambda: \mathcal{A} \rightarrow \mathfrak{zh}_P$ is an appropriate linear map vanishing on scalars. Now applying (2.30) and performing elementary transformations we find

$$\begin{aligned} \int_P D(\varphi) &= \int_P \nabla(\varphi) + (-)^{\partial\varphi} \sum_k \int_P [\varphi_k \lambda(c_k)] = (-)^{\partial\varphi} \sum_k \int_P [\varphi_k \lambda(c_k^{(1)}) h(c_k^{(2)})] \\ &= (-)^{\partial\varphi} \sum_k \int_P [\varphi_k h(c_k)] \lambda(1) = 0. \quad \square \end{aligned}$$

In particular, this property implies that every bundle derivative will be formally adjointable, with the formal adjoint given by

$$(5.2) \quad D^\dagger(\varphi) = -\star_P^{-1} D\star_P(\varphi).$$

Further proceeding with this, it is possible to write the explicit formulas for the Dirac operator and the Laplacian associated to an arbitrary bundle derivative D . As explained in [5] the system of bundle derivatives allows us to construct a natural bicovariant \ast -calculus Γ on the structure group G , and a natural calculus on the bundle (extending the algebra \mathfrak{hor}_P of horizontal forms by taking a cross product with the appropriate braided exterior algebra Γ_{inv}^\wedge). We can also extend the whole picture to the level of $\mathfrak{hor}_{P,\Sigma}$ and $\Omega_{M,\Sigma}$.

5.2. Quantum Hopf Fibration

This highly instructive example shows us that the asymptotics of the eigenvalues of the Dirac operator could be quite surprising in the non-commutative context. We are going to deal with the Dirac operator over a quantum 2-sphere. We refer to [13] for detailed calculations.

By definition, the quantum Hopf fibration is a quantum $U(1)$ -bundle over a quantum 2-sphere [16]. The total space of the bundle is given by the quantum $SU(2)$ -group. In other words [19], the bundle \ast -algebra \mathcal{B} is generated by two elements $\{\alpha, \gamma\}$ and the following relations:

$$\begin{aligned} \alpha\alpha^* + \mu^2\gamma\gamma^* &= 1 & \alpha^*\alpha + \gamma^*\gamma &= 1 \\ \alpha\gamma &= \mu\gamma\alpha & \alpha\gamma^* &= \mu\gamma^*\alpha & \gamma\gamma^* &= \gamma^*\gamma, \end{aligned}$$

where $\mu \in [-1, 1] \setminus \{0\}$. On the other hand, the Hopf \ast -algebra \mathcal{A} of the structure group $G = U(1)$ is generated by a single unitary element U . The coproduct is specified by $\phi(U) = U \otimes U$.

It is also worth mentioning that the matrix

$$u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

defines the fundamental representation of the quantum $SU(2)$ group.

The above relations defining \mathcal{B} are equivalent to the unitarity property $u^{-1} = u^*$. The coproduct map $\phi: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ is uniquely determined by $\phi(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. We have denoted by the same symbol for the coproducts on G and P .

Our structure group G is understandable as a *subgroup* of P , in accordance with the identification

$$\mathcal{A} \leftrightarrow \mathcal{B}/\text{gen}(\gamma, \gamma^*).$$

If $\mu \neq 1, -1$ then G is exactly the classical part of P . The action $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ is constructed from the coproduct, by taking the factor-projection on the second tensorand, in other words $F = (\text{id} \otimes [\cdot]_{\mathcal{A}})\phi$. The algebra \mathcal{V} describing the quantum 2-sphere M is defined as the F -fixed point subalgebra of \mathcal{B} , so that the map i is just the inclusion.

Let us now sketch the construction of a canonical frame structure on the quantum 2-sphere [7]. We shall start from the canonical 3-dimensional left-covariant and \ast -covariant calculus Φ over P . This calculus is constructed in [19]. Explicitly the space Φ_{inv} is spanned by the elements

$$\eta_3 = \pi(\alpha - \alpha^*) \quad \eta_+ = \pi(\gamma) \quad \eta_- = \pi(\gamma^*)$$

and the canonical right \mathcal{B} -module structure \circ on Φ_{inv} is given by

$$\begin{aligned} \mu^2 \eta_3 \circ \alpha &= \eta_3 & \eta_3 \circ \alpha^* &= \mu^2 \eta_3 & \Phi_{inv} \circ \gamma &= \Phi_{inv} \circ \gamma^* = \{0\}. \\ \mu \eta_{\pm} \circ \alpha &= \eta_{\pm} & \eta_{\pm} \circ \alpha^* &= \mu \eta_{\pm} \end{aligned}$$

This \mathcal{B} -module structure on Φ_{inv} factorizes through the ideal $\text{gen}\{\gamma, \gamma^*\}$ and induces a right \mathcal{A} -module structure on the same space (and will be denoted by the same symbol). Let \mathbb{V} be a vector space spanned by η_{\pm} . It will be equipped with the constructed \circ and $*$ -structures, and we shall assume that

$$(5.3) \quad \chi(\eta_+) = \eta_+ \otimes U^2 \quad \chi(\eta_-) = \eta_- \otimes U^{-2}.$$

Such a definition allows us to interpret $\chi: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathcal{A}$ as *the adjoint* action of G , coming from the group structure in P . It follows that (in the basis η_{\pm}) the associated braid operator $\tau: \mathbb{V}^{\otimes 2} \rightarrow \mathbb{V}^{\otimes 2}$ looks like

$$(5.4) \quad \tau = \begin{pmatrix} 1/\mu^2 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu^2 & 0 \\ 0 & \mu^2 & 0 & 0 \\ 0 & 0 & 0 & \mu^2 \end{pmatrix}$$

The corresponding τ -exterior algebra is given by the relations

$$(5.5) \quad \eta_{\pm}^2 = 0 \quad \eta_+ \eta_- = -\mu^2 \eta_- \eta_+.$$

It is worth noticing that these relations are a subset of the relations defining the canonical higher-order calculus over P , given by the universal differential envelope Φ^{\wedge} of Φ ([20],[3]–Appendix B). For completeness, we shall list here the remaining relations (involving η_3). These are

$$(5.6) \quad \eta_3^2 = 0, \quad \eta_3 \eta_{\pm} = \mu^{\mp 4} \eta_{\pm} \eta_3.$$

This means that \mathfrak{hor}_P is viewable as a subalgebra of Φ^{\wedge} generated by $\mathcal{B} = \mathfrak{hor}_P^0$ and the elements η_+, η_- . Furthermore, we can introduce a natural projection homomorphism $p_{hor}: \Phi^{\wedge} \rightarrow \mathfrak{hor}_P$, defined by

$$p_{hor}(\eta_3) = 0 \quad p_{hor} \upharpoonright \mathfrak{hor}_P = \text{id}.$$

The canonical antiderivation $\nabla: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ is defined as the composition of this projection with the differential $d: \Phi^{\wedge} \rightarrow \Phi^{\wedge}$.

The constructed map coincides with the covariant derivative of the canonical regular connection introduced in [4]. It corresponds to the standard Levi-Civita connection on the 2-sphere.

Let us observe that we are already in the context of the spin bundles. The analog of the orthonormal frame bundle P_0 over the 2-sphere M is given by the $*$ -subalgebra $\mathcal{B}_+ \subseteq \mathcal{B}$ corresponding to the quantum $\text{SO}(3)$ group (even combinations of generators $\alpha, \gamma, \gamma^* \alpha^*$). The structure group $G = \text{U}(1)$ is here understood as a 2-fold covering of the structure group $G_0 = \text{SO}(2)$ of P_0 .

A braid operator σ is given by the matrix

$$(5.7) \quad \sigma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1/\mu^2 & 0 \\ 0 & \mu^2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and the $*$ -structure on \mathbb{V} is specified by $\eta_+^* = \mu\eta_-$, while $\eta_3^* = -\eta_3$. The quantum metric is defined by

$$(5.8) \quad g_{+-}^* = g_{+-} = \mu^2 g_{-+} \quad g_{++} = g_{--} = 0.$$

The algebra Σ is infinite-dimensional. A realization of Σ in the Hilbert space $H = l^2(\mathbb{Z})$ is given by

$$(5.9) \quad g_{+-} : e_k \mapsto \frac{1}{2} \mu^{2k} e_k$$

where $\{e_k \mid k \in \mathbb{Z}\}$ are canonical basis vectors in H . A common domain H_0 for all the operators from Σ consists of sequences with finite support, for example.

In what follows we shall assume that the deformation parameter is positive (the case $\mu < 0$ gives the same results). For the spinor space, we shall take $\mathbb{S} = \mathbb{C}^2$ with the canonical basis $|+\rangle, |-\rangle$ and the action

$$\kappa_{\mathbb{S}}|+\rangle = |+\rangle \otimes U \quad \kappa_{\mathbb{S}}|-\rangle = |-\rangle \otimes U^{-1}.$$

The metric g and the algebra Σ are completely determined by the assignment

$$(5.10) \quad \eta_+ \mapsto \mu^{1/2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \eta_- \mapsto \mu^{-1/2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

in particular it follows that

$$(5.11) \quad \frac{1}{\mu} \gamma[g_{+-}] = \mu \gamma[g_{-+}] = \frac{1}{2} \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix} \quad \gamma[\Sigma] = \mathbb{C} \oplus \mathbb{C}.$$

We see that the spinor representation $\gamma : \Sigma \rightarrow L(\mathbb{S})$ is not faithful.

By the way, it is easy to see that all possible irreducible representations of this Clifford algebra (with fixed spectral properties of Σ) are given by

$$\gamma[\eta_+] = \mu^{1/2+k} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \gamma[\eta_-] = \mu^{-1/2+k} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where $k \in \mathbb{Z}$.

We are now going to study the associated Dirac operator. For each spin level $s \in \mathbb{N}/2$ let u^s be the matrix of the canonical spin- s irreducible representation of P . The matrix elements of all possible representations u^s form a natural basis in \mathcal{B} . Let us denote by \mathcal{B}_s the subspace of \mathcal{B} spanned by the matrix elements of u^s . We have

$$\mathcal{B} = \mathbb{C} \oplus \sum_{s \in \mathbb{N}/2}^{\oplus} \mathcal{B}_s.$$

By construction, the coordinate vector fields ∂_-, ∂_+ of the frame structure coincide with the spin creation and annihilation operators iK_{\mp} for the right regular representation, given by the coproduct $\phi : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$. Obviously $\phi(\mathcal{B}_s) \subseteq \mathcal{B}_s \otimes \mathcal{B}_s$, and moreover the space \mathcal{B}_s is characterized as the multiple irreducible spin- s subspace of \mathcal{B} . In terms of the matrix elements u_{ij}^s , the operators ∂_{\pm} act nontrivially only on the second indexes, while the first indexes are ‘free’.

Therefore, we can write

$$\mathcal{B}_s \leftrightarrow \underbrace{H_s \oplus \cdots \oplus H_s}_{2s+1}$$

and introduce a canonical basis in \mathcal{B}_s of the form $\{\psi_{\alpha s}^m \mid m, \alpha = -s, \dots, s\}$. Here α is interpreted as a ‘degeneration index’.

In summary, we have

$$\begin{aligned} K_+(\psi_{\alpha s}^m) &= -i\partial_+(\psi_{\alpha s}^m) = \frac{1}{\mu^{s+m}} \left\{ (s-m)_\mu (s+m+1)_\mu \right\}^{1/2} \psi_{\alpha s}^{m+1} \\ K_-(\psi_{\alpha s}^m) &= -i\partial_-(\psi_{\alpha s}^m) = \frac{\mu}{\mu^{s+m}} \left\{ (s-m+1)_\mu (s+m)_\mu \right\}^{1/2} \psi_{\alpha s}^{m-1} \\ F(\psi_{\alpha s}^m) &= \psi_{\alpha s}^m \otimes U^m \quad n_\mu = \frac{1-\mu^{2n}}{1-\mu^2}. \end{aligned}$$

Hence the spinor module \mathcal{S} is decomposed as follows

$$(5.12) \quad \mathcal{S} = \sum_{s \in \mathbb{N}-1/2}^{\oplus} \mathcal{S}_s,$$

where the spaces \mathcal{S}_s are spanned by vectors

$$\psi_{\alpha s}^{1/2} \otimes |-\rangle \quad \psi_{\alpha s}^{-1/2} \otimes |+\rangle$$

with the degeneracy index α arbitrary.

The Dirac operator is given by

$$i\mathbb{D} = \partial_+ \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \partial_- \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and it follows that $\mathbb{D}(\mathcal{S}_s) \subseteq \mathcal{S}_s$ for each $s \in \mathbb{N} - 1/2$. It is now very easy to diagonalize the reduced operators. We have two eigenvalues with the eigenvectors of the form

$$(5.13) \quad \mathbb{D}\varphi_{\alpha s}^+ = \lambda_s \varphi_{\alpha s}^+ \quad \mathbb{D}\varphi_{\alpha s}^- = -\lambda_s \varphi_{\alpha s}^-,$$

where

$$(5.14) \quad \lambda_s = \frac{\mu^{s+1/2} - \mu^{-s-1/2}}{\mu - \mu^{-1}}.$$

It is worth noticing that the eigenvalues satisfy the following recurrent formula

$$(5.15) \quad \lambda_s = \frac{\lambda_{s+1} + \lambda_{s-1}}{\mu + \mu^{-1}}.$$

In the above formulas we assumed that $\mu > 0$. The spectrum is invariant if we replace $\mu \mapsto -\mu$. On the other hand, if $\mu = 1, -1$ the spectrum will be given by $\lambda_s = 2s$. The case $\mu = -1$ is very special, as it gives us a quantum spin bundle over the classical 2-sphere M . It illustrates a general phenomenon that the classification problem of quantum principal bundles is qualitatively different from its classical counterpart, even if both the base manifold and the structure group are classical.

5.3. Discussion of Quantum Phenomena

In classical geometry a very important geometrical information is contained in the eigenvalues of standard elliptic operators. In particular, the distribution of eigenvalues of the Dirac operator reflects the deepest structure of compact Riemannian spin manifolds.

If we look at the asymptotics of the spectrum of (the modulus of) the Dirac operator over the quantum 2-sphere (with their degeneracies taken into account) we arrive to the following expression

$$(5.16) \quad a_N \sim \begin{cases} \sqrt{N} & \text{for } \mu = \pm 1, \\ \mu^{-\sqrt{N/2}} & \text{if } \mu \in (0, 1) \end{cases}$$

with the symmetry $\mu \mapsto -\mu$. In particular, we see that the inverse of the Dirac operator is *trace class* in the fully quantum case $\mu \neq 1, -1$. This is not compatible with the formulation proposed in [2], where it was assumed that the quantum Dirac operator will always have a similar asymptotics as in the classical geometry

$$a_N \sim N^{1/d}, \quad d = \dim(M).$$

Quantum geometry gives us much more freedom, and it is not possible to cover the diversity of all possible quantum spaces by a single asymptotic expression.

In our theory (as far as we consider the pure Riemannian geometry) the group G plays the role of special orthogonal structure group $SO(d)$, and \varkappa plays the role of its fundamental representation. In various interesting examples the representation $\varkappa: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathcal{A}$ will be irreducible. However, in general this representation will be reducible. The fact that $\Sigma \neq \mathbb{C}$ allows us to overcome inherent obstacles that would appear in the formalism, in the case of irreducible $\varkappa: \mathbb{V} \rightarrow \mathbb{V} \otimes \mathcal{A}$.

This is due to the fact that the braid operators $\tau, \sigma: \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$, the $*$ -structure and the quadratic form on \mathbb{V} are all \varkappa -covariant. If the metric g would take values from \mathbb{C} then using elementary algebraic operations with $g, *$ and σ, τ we would be able to build various intertwiners of \varkappa out of these objects. In general, these intertwiners would be non-scalar operators, which implies that \varkappa will be reducible. The non-triviality of Σ overcomes this obstacle.

Another interesting quantum phenomenon is that the grade m of the volume form w is not necessarily the same as the number d of coordinate one-forms.

Our definition of a quantum Clifford algebra is motivated by considerations presented in [14] and [11], based on deformations of braided exterior algebras. Our main condition is similar—the vector space \mathbb{V}^\wedge is equipped with a new product however this new product is a quantization of \wedge with a non-commutative deformation parameter. Of course, this is connected with a non-scalar nature of metric components $g_{ij}: \mathbb{S} \rightarrow \mathbb{S}$.

Finally, let us mention that among possible algebraic conditions involving braid operators τ and σ , there exist natural equations [10] closely related to multiplicative unitaries [23] associated to compact quantum groups.

APPENDIX A. SIMPLE C^* -ALGEBRAIC CONSIDERATIONS

In this appendix we shall consider the vertical integration map, and the associated GNS-type construction. Let $P = (\mathcal{B}, i, F)$ be an arbitrary quantum principal G -bundle over M .

Let us consider the map $\int_\dagger: \mathcal{B} \rightarrow \mathcal{V}$ defined by $i[\int_\dagger(b)] = (\text{id} \otimes h)F(b)$. We shall assume that \mathcal{V} is realized as an everywhere dense $*$ -subalgebra of a unital C^* -algebra $\widehat{\mathcal{V}}$.

One of the principal properties we have considered in the main text was the strict positivity condition. We shall first prove that this condition will be automatically satisfied in certain important special cases.

Proposition A.1. *Let us assume that \mathcal{B} admits a C^* -algebraic closure $\widehat{\mathcal{B}}$ such that $i: \mathcal{V} \rightarrow \mathcal{B}$ extends to a $*$ -monomorphism $i: \widehat{\mathcal{V}} \rightarrow \widehat{\mathcal{B}}$. Let us assume that \mathcal{V} is stable in $\widehat{\mathcal{V}}$, under holomorphic functional calculus. Then the vertical integration map \int_\dagger is strictly positive.*

Proof. Let us start from the canonical [20] orthogonality relations

$$(A.1) \quad h[u_{ki}^* u_{lj}] = \delta_{ij} [C_u^{-1}]_{lk} / \text{tr}(C_u)$$

$$(A.2) \quad h[u_{ki} u_{lj}^*] = \delta_{kl} [C_u]_{ji} / \text{tr}(C_u^{-1})$$

where u is an arbitrary unitary irreducible matrix representation of G , and C_u is the canonical intertwiner between u and its second contragradient u^{cc} .

In order to verify the positivity property, it is sufficient to consider only the elements from a fixed multiple irreducible submodule \mathcal{B}^u .

A general element of such a form is given by

$$b = \sum_{i\alpha} f_{i\alpha} b_{\alpha i} = \text{tr}[\Phi B]$$

where $f_{i\alpha} \in \mathcal{V}$ form a matrix Φ while $b_{\alpha i}$ form a matrix B and satisfy

$$(A.3) \quad \sum_{\alpha} b_{\alpha i}^* b_{\alpha j} = \delta_{ij} 1 \quad F(b_{\alpha i}) = \sum_j b_{\alpha j} \otimes u_{ji}.$$

This is a consequence of the holomorphic stability of \mathcal{V} in $\widehat{\mathcal{V}}$. Now a direct computation gives

$$\begin{aligned} \int_{\uparrow} (bb^*) &= \sum_{ij\alpha\beta} f_{i\alpha} \int_{\uparrow} (b_{\alpha i} b_{\beta j}^*) f_{j\beta}^* = \sum_{ijpq\alpha\beta} f_{i\alpha} b_{\alpha p} b_{\beta q}^* h(u_{pi} u_{qj}^*) f_{j\beta}^* \\ &= \sum_{ijpq\alpha\beta} f_{i\alpha} b_{\alpha p} b_{\beta q}^* f_{j\beta}^* \delta_{pq} [C_u]_{ji} / \text{tr}[C_u^{-1}] \\ &= \sum_{ij\alpha\beta} f_{i\alpha} (BB^\dagger)_{\alpha\beta} f_{j\beta}^* [C_u]_{ji} / \text{tr}[C_u^{-1}] = \text{tr} \left\{ C_u \Phi B B^\dagger \Phi^\dagger \right\} / \text{tr}[C_u^{-1}] \end{aligned}$$

and from this expression it follows that \int_{\uparrow} is really strictly positive. \square

Let us now assume that the bundle P is such that \int_{\uparrow} is strictly positive.

Proposition A.2. *This property of \int_{\uparrow} implies the existence of a canonical faithful GNS-type representation of \mathcal{B} , which enables us to introduce a C^* -norm in \mathcal{B} .*

Proof. The strict positivity property implies that we can introduce a pre-Hilbert C^* -module \mathcal{B} , with the \mathcal{V} -valued scalar product

$$\langle b, q \rangle_M = \int_{\uparrow} (b^* q).$$

The multiple irreducible subspaces \mathcal{B}^u will be orthogonal relative to \langle, \rangle_M . Let us denote by \mathcal{H} the completion of this module. The multiplication map in \mathcal{B} naturally induces a $*$ -homomorphism $D: \mathcal{B} \rightarrow \mathbb{B}(\mathcal{H})$. Here $\mathbb{B}(\mathcal{H})$ is the C^* -algebra of bounded right $\widehat{\mathcal{V}}$ -linear adjointable operators in \mathcal{H} . In order to prove the boundness of the operators $D[\cdot]$ it is sufficient to check it on canonical multiplets $b_{\alpha i}$ and on the elements $f \in \mathcal{V}$. We have

$$\begin{aligned} \langle D[b_{\alpha i}] \psi, D[b_{\alpha i}] \psi \rangle_M &\leq \sum_{\beta} \langle D[b_{\beta i}] \psi, D[b_{\beta i}] \psi \rangle_M = \sum_{\beta} \langle \psi, D[b_{\beta i}^* b_{\beta i}] \psi \rangle_M \\ &= \langle \psi, \psi \rangle_M \end{aligned}$$

and it follows that $D[b_{\alpha i}]$ are bounded with $\|D[b_{\alpha i}]\| \leq 1$. It is easy to verify that $D: \mathcal{V} \rightarrow \mathbb{B}(\mathcal{H})$ is isometric.

Obviously, the unit element $1 \in \mathcal{H}$ is a cyclic and separating vector for D . In particular D is faithful. \square

The constructed C*-norm has some further interesting properties. At first, let us observe that

$$(A.4) \quad \left| \int_{\uparrow} b \right| \leq |b|$$

for each $b \in \mathcal{B}$. In particular, the vertical integration map extends continuously to the C*-completion $\widehat{\mathcal{B}}$ and the above inequality holds on the whole $\widehat{\mathcal{B}}$. Secondly, the C*-structure on \mathcal{B} is unique. Indeed, let $||_{\max}$ be the universal C*-norm on \mathcal{B} extending the C*-norm on \mathcal{V} , and let B_{\max} be the corresponding C*-completion. The existence of such a maximal C*-norm follows easily from equalities (A.3) and the existence of D . By construction F extends by continuity to a *-homomorphism $F: B_{\max} \rightarrow B_{\max} \otimes A$. It is easy to see that $\omega = (\text{id} \otimes h)F: B_{\max} \rightarrow \widehat{\mathcal{V}}$ is strictly positive (assuming that the Haar measure on G is faithful) and $|\omega(b)| \leq |b|_{\max}$ on B_{\max} . This implies that $B_{\max} = \widehat{\mathcal{B}}$. In particular $||_{\max} = ||$ and $\omega = \int_{\uparrow}$.

The map $F: \widehat{\mathcal{B}} \rightarrow \widehat{\mathcal{B}} \otimes A$ satisfies

$$\begin{array}{ccc} \widehat{\mathcal{B}} & \xrightarrow{F} & \widehat{\mathcal{B}} \otimes A \\ F \downarrow & & \downarrow \text{id} \otimes \phi \\ \widehat{\mathcal{B}} \otimes A & \xrightarrow{F \otimes \text{id}} & \widehat{\mathcal{B}} \otimes A \otimes A \end{array} \quad \widehat{\mathcal{B}} \otimes A = \overline{\text{lin}\{F(b)a\}}$$

which are purely C*-algebraic counterparts of the quantum group action axioms. The second equality expresses the idea that the group G acts by homeomorphisms on the bundle space P . We conclude this appendix by

$$\widehat{\mathcal{B}} \otimes A = \overline{\text{lin}\{qF(b) \mid q, b \in \widehat{\mathcal{B}}\}}$$

which is a purely C*-algebraic version of the freeness axiom.

APPENDIX B. BRAIDED CLIFFORD ALGEBRAS

In this Appendix we present an introduction to a general theory of braided Clifford algebras and spinors. Much of the material is logically independent of our main context of principal bundles, quantum groups and bicovariant bimodules.

Let us consider a complex finite-dimensional vector space \mathbb{V} equipped with a regular braid operator $\sigma: \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$ and an antilinear involution $*$: $\mathbb{V} \rightarrow \mathbb{V}$. The involution map extends naturally to a *-structure on the tensor algebra \mathbb{V}^{\otimes} . Explicitly,

$$(\xi_1 \otimes \cdots \otimes \xi_n)^* = \xi_n^* \otimes \cdots \otimes \xi_1^*.$$

We have denoted by the same symbol $*$ a unique antimultiplicative (unital) antilinear extension on the tensor algebra. We shall assume that

$$(B.1) \quad *\sigma = \sigma*.$$

First, we are going to formalize the idea of a *quantum metric*. As we already mentioned it will be allowed that ‘metric coefficients’ do not commute.

Let Σ be a *-algebra, and let $g: \mathbb{V} \otimes \mathbb{V} \rightarrow \Sigma$ be a linear map. Below we have listed a number of interesting identities involving g , $*$ and σ :

(i) **Braided-symmetry of the metric.** In other words

$$(B.2) \quad g\sigma = g.$$

For this to make any sense, it is necessary that 1 belongs to the spectrum of σ .

(ii) **Reality property.** We have

$$(B.3) \quad g(x, y)^* = g(y^*, x^*) \quad \forall x, y \in \mathbb{V}$$

(iii) **Funny σ -compatibility.** We have

$$(B.4) \quad \begin{aligned} g \otimes_{\Sigma} g &= (g \otimes_{\Sigma} g)(\text{id} \otimes \sigma \otimes \text{id})(\sigma^{-1} \otimes \sigma)(\text{id} \otimes \sigma^{-1} \otimes \text{id}) \\ g \otimes_{\Sigma} g &= (g \otimes_{\Sigma} g)(\text{id} \otimes \sigma \otimes \text{id})(\sigma \otimes \sigma^{-1})(\text{id} \otimes \sigma^{-1} \otimes \text{id}). \end{aligned}$$

Let us observe that the above two equations are *equivalent* if we assume that reality condition holds. As we shall see, this property ensures that g is extendible, to the level of appropriate Σ -bimodules.

(iv) **Weak positivity.** In order to formulate this property we have to assume that Σ is realized by operators in the Hilbert space $H = l^2(\mathbb{Z})$. Since in general these operators will be *unbounded* we have to take care about the domains. We shall assume that there is an everywhere dense linear subspace $H_0 \subseteq H$ which is a common domain for all the operators from Σ . We shall also assume that the $*$ -structure on Σ is represented as taking formal adjoints of linear operators in H_0 , and that there exists cyclic and separating vectors $\Omega \in H_0$ for Σ .

So one natural version of positivity would be

$$(B.5) \quad g(x^*, x) \geq 0 \quad x = 0 \Leftrightarrow g(x^*, x) = 0 \quad \forall x \in \mathbb{V}.$$

The reason why we call this condition ‘weak’ positivity will become clear after we construct a canonical Σ -bimodule structure over \mathbb{V} , and introduce a stronger version of positivity.

(v) **Minimality & Invertibility.** The matrix $g_{ij} = g(\theta_i, \theta_j)$ is invertible in Σ , where $\{\theta_1, \dots, \theta_d\}$ are basis vectors in \mathbb{V} . Moreover, the algebra Σ is generated by the matrix elements of g and g^{-1} .

(vi) **Twisting Σ and \mathbb{V} .** Let $\nu_{\Sigma}: \Sigma \rightarrow M_d(\Sigma)$ be a unital homomorphism. Obviously, this map is completely determined by its values on the elements g_{ij} and it gives us the structure of a right Σ -module, in the free left Σ -module $\mathbb{V}_{\Sigma} \leftrightarrow \Sigma \otimes \mathbb{V}$, so that we have a bimodule structure. The right Σ -multiplication is simply given by

$$\theta_i q = \sum_j \nu_{\Sigma}(q)_{ij} \theta_j.$$

Equivalently, the map ν_{Σ} can be viewed as a twisting operator

$$\nu_{\Sigma}: \mathbb{V} \otimes \Sigma \rightarrow \Sigma \otimes \mathbb{V} \quad \nu_{\Sigma}(\theta_i \otimes q) = \sum_j \nu_{\Sigma}(q)_{ij} \otimes \theta_j.$$

This twisting preserves the product and the unit in Σ , in a natural way.

We shall assume that

$$(B.6) \quad \nu_{\Sigma}(\text{id} \otimes g) = (g \otimes \text{id})(\text{id} \otimes \sigma)(\sigma^{-1} \otimes \text{id}),$$

which completely fixes ν_{Σ} .

Lemma B.1. *The $*$ -involutions on \mathbb{V} and Σ naturally combine to a $*$ -structure on the bimodule \mathbb{V}_{Σ} . In particular the map $\nu_{\Sigma}: \mathbb{V} \otimes \Sigma \rightarrow \Sigma \otimes \mathbb{V}$ is invertible and*

$$(B.7) \quad {}^* \nu_{\Sigma} {}^* = \nu_{\Sigma}^{-1}.$$

In particular, it follows that \mathbb{V}_{Σ} free, as a right Σ -module.

Proof. The map $*$: $\mathbb{V}_\Sigma \rightarrow \mathbb{V}_\Sigma$ is introduced by $(q \otimes \theta_j)^* = \theta_j^* q^*$. It is sufficient to prove that such a map is involutive. This follows easily from (B.7) and the reality properties for σ and g . \square

More generally, it is easy to see that

$$\mathbb{V}_\Sigma^{\otimes n} \leftrightarrow \overbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}^n \otimes \Sigma \leftrightarrow \Sigma \otimes \underbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}_n$$

in a natural manner, as a right/left Σ -module.

Let us observe that our definition of the Σ -bimodule structure implies that the braiding $\sigma: \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$ is (necessarily uniquely) extendible to a bimodule homomorphism $\sigma: \mathbb{V}_\Sigma^{\otimes 2} \rightarrow \mathbb{V}_\Sigma^{\otimes 2}$. Similarly, property (B.4) ensures that the metric tensor $g: \mathbb{V} \otimes \mathbb{V} \rightarrow \Sigma$ is uniquely extendible to a Σ -bilinear map $g: \mathbb{V}_\Sigma^{\otimes 2} \rightarrow \Sigma$. In playing with such extended maps, it is useful to recall the following simple lemma:

Lemma B.2. *Let \mathcal{M} and \mathcal{N} be left Σ -modules and let $\Phi: \mathcal{M} \rightarrow \mathcal{N}$ be a linear map satisfying*

$$\Phi(g_{ij}\xi) = g_{ij}\Phi(\xi).$$

Then Φ is left Σ -linear.

Proof. It is sufficient to check that Φ commutes with left multiplications by the matrix elements $[g^{-1}]_{ij}$. However this is equivalent to the above formula. \square

Our extensions preserve all relevant algebraic relations between g , σ and $*$. There is an interesting way to describe the relation between the left and the right Σ -module structures on \mathbb{V}_Σ . They are related by

$$(B.8) \quad (\text{id} \otimes g)(\sigma \otimes \text{id}) = (g \otimes \text{id})(\text{id} \otimes \sigma).$$

In this formula the tensor product is taken over Σ .

The space \mathbb{V}_Σ , together with the extended σ , generates a braided monoidal category \mathcal{C} . We shall use the same symbol σ to denote the generic braiding in this category. Moreover, we shall use the same symbol g for extended contraction maps $g: \mathbb{V}_\Sigma^{\otimes n} \otimes_\Sigma \mathbb{V}_\Sigma^{\otimes n} \rightarrow \Sigma$. The extended maps are defined inductively by

$$(B.9) \quad g\{(\psi \otimes x) \otimes (y \otimes \xi)\} = g(\psi, g(x, y)\xi),$$

where $x, y \in \mathbb{V}$ and we shall also assume that tensors with different grades are mutually ‘orthogonal’.

Let us now consider a map $\langle \rangle: \mathbb{V}_\Sigma \times \mathbb{V}_\Sigma \rightarrow \Sigma$ defined by:

$$(B.10) \quad \langle \psi, \xi \rangle = g(\psi^*, \xi).$$

It is easy to see that the following identities are fulfilled:

$$(B.11) \quad \langle \psi, \xi a \rangle = \langle \psi, \xi \rangle a \quad \langle \psi a, \xi \rangle = a^* \langle \psi, \xi \rangle$$

$$(B.12) \quad \langle \psi, a\xi \rangle = \langle a^* \psi, \xi \rangle$$

$$(B.13) \quad \langle \psi, \xi \rangle^* = \langle \xi, \psi \rangle$$

$$(B.14) \quad \langle \psi, \xi + \varphi \rangle = \langle \psi, \xi \rangle + \langle \psi, \varphi \rangle.$$

This map plays the role of a hermitian scalar product in \mathbb{V}_Σ . To completely justify this interpretation, it is necessary to formulate the appropriate positivity condition.

(vii) **Strict positivity.** Assuming that Σ is realized in $H = l^2(\mathbb{Z})$ we have

$$(B.15) \quad \langle \xi, \xi \rangle = g(\xi^*, \xi) > 0, \quad \forall \xi \in \mathbb{V}_\Sigma \setminus \{0\}.$$

This condition is, in general, stronger than (iv). It is easy to construct examples where (iv) holds and (vii) fails. It is easy to see that the Σ -valued scalar product is naturally extendible to higher-order tensor blocks $\mathbb{V}_\Sigma^{\otimes n}$. All algebraic properties are preserved.

In what follows, it will be assumed that conditions (i)–(iii) and (v)–(vii) are satisfied. In particular, we see that \mathbb{V}_Σ equipped with $\langle \rangle$ gives us a (generally unbounded) unitary bimodule over Σ . Furthermore, the extended scalar products on spaces $\mathbb{V}_\Sigma^{\otimes n}$ are understandable as n -fold tensor iterations of the initial bimodule \mathbb{V}_Σ . In particular, it follows that all extended $\langle \rangle$ are strictly positive, too (all $\mathbb{V}_\Sigma^{\otimes n}$ are unitary bimodules).

Lemma B.3. *The braid operator $\sigma: \mathbb{V}_\Sigma^{\otimes 2} \rightarrow \mathbb{V}_\Sigma^{\otimes 2}$ is hermitian. In other words, we have*

$$(B.16) \quad \langle \psi, \sigma(\xi) \rangle = \langle \sigma(\psi), \xi \rangle \quad \forall \psi, \xi \in \mathbb{V}_\Sigma^{\otimes 2}.$$

In particular, the map $\sigma: \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \otimes \mathbb{V}$ is diagonalizable, and has real eigenvalues.

Proof. The hermicity property follows from (B.1) and

$$(B.17) \quad g(\psi, \sigma[\xi]) = g(\sigma[\psi], \xi) \quad \forall \psi, \xi \in \mathbb{V} \otimes \mathbb{V}$$

which, in its turn, follows from the definition of the bimodule structure on \mathbb{V}_Σ . Let $\omega: \Sigma \rightarrow \mathbb{C}$ be an arbitrary faithful state on Σ . Then $\omega\langle \rangle$ is a scalar product on $\mathbb{V} \otimes \mathbb{V}$, and σ is hermitian with respect to this scalar product. \square

We are now ready to construct and study braided Clifford algebras. Conceptually, we follow [11] which means that our Clifford algebras will be understood as Chevalley-Kahler-type deformations of braided exterior algebras.

Let \mathbb{V}^\wedge be the braided exterior algebra [22] built over (σ, \mathbb{V}) . This algebra is defined as

$$\mathbb{V}^\wedge = \mathbb{V}^\otimes / \ker(A_\sigma)$$

where A_σ is the total braided antisymmetrizer map. The exterior algebra gets its $*$ -structure from \mathbb{V}^\otimes . Furthermore, the algebra \mathbb{V}^\wedge possess a natural braided Hopf algebra structure, where the coproduct map is specified by

$$(B.18) \quad \phi(\alpha) = \sum_{i=0}^n B_{in-i}(\alpha) \quad \alpha \in \mathbb{V}^{\wedge n}$$

and $B_{kl}: \mathbb{V}^{\wedge k+l} \rightarrow \mathbb{V}^{\wedge k} \otimes \mathbb{V}^{\wedge l}$ is the corresponding braided inverse-shuffle operator.

The coproduct map has a particularly elegant form if we make natural identifications

$$\mathbb{V}^{\wedge n} \leftrightarrow \text{im}(A_\sigma^n)$$

induced by the antisymmetrizer map. In terms of these identifications, we have

$$(B.19) \quad \phi(\alpha) = \sum_{i=0}^n \{x_1 \otimes \cdots \otimes x_i\} \otimes \{x_{i+1} \otimes \cdots \otimes x_n\}$$

with $\alpha = \sum x_1 \otimes \cdots \otimes x_n$. The antipode map is braided-antimultiplicative (acting as a total σ -inverse permutation). The Hopf algebra structure is compatible with the $*$ -involution, in the sense that

$$\phi(\alpha^*) = \phi(\alpha)^* \quad \kappa(\alpha^*) = \kappa(\alpha)^*.$$

All considerations with the braided exterior algebra are incorporable to the level of Σ -modules. More precisely, let \mathbb{V}_Σ^\wedge be the braided exterior algebra constructed from \mathbb{V}_Σ and the extended σ . We have the following natural identifications

$$\mathbb{V}_\Sigma^\wedge \leftrightarrow \mathbb{V}^\wedge \otimes \Sigma \leftrightarrow \Sigma \otimes \mathbb{V}^\wedge$$

of right/left Σ -modules. The coproduct map ϕ is naturally (and uniquely) extendible to a Σ -bilinear map $\phi: \mathbb{V}_\Sigma^\wedge \rightarrow \mathbb{V}_\Sigma^\wedge \otimes_\Sigma \mathbb{V}_\Sigma^\wedge$. In a similar way, it is possible to extend the coinverse and the counit.

Observe now that the block antisymmetrizers $A_\sigma^n: \mathbb{V}^{\otimes n} \rightarrow \mathbb{V}^{\otimes n}$ are hermitian maps, and commute with the $*$ -structure. Hence the pairing $g_\wedge: \mathbb{V}_\Sigma^{\otimes} \times \mathbb{V}_\Sigma^{\otimes} \rightarrow \Sigma$ defined by

$$g_\wedge(\psi, \xi) = g(\psi, A_\sigma \xi) \quad g_\wedge(1, 1) = 1$$

is projectable down to a map $g_\wedge: \mathbb{V}_\Sigma^\wedge \times \mathbb{V}_\Sigma^\wedge \rightarrow \Sigma$.

Our braided Clifford algebra $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ will be identified with \mathbb{V}_Σ^\wedge at the level of Σ -bimodules. The $*$ -structure will also be the same. However, $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ will be equipped with a *new product* defined by

$$(B.20) \quad \tilde{m} = m(\text{id} \otimes g_\wedge \otimes \text{id})(\phi \otimes \phi),$$

where $m: \mathbb{V}_\Sigma^\wedge \otimes \mathbb{V}_\Sigma^\wedge \rightarrow \mathbb{V}_\Sigma^\wedge$ is the original product in \mathbb{V}_Σ^\wedge . In the above formula, we have adopted the standard *Cliffordization procedure* from the classical theory [17]. The first thing to examine is that we really obtain a nice $*$ -algebra this way.

Proposition B.4. *The product \tilde{m} is associative and 1 is the unit element. Moreover, the $*$ -involution is \tilde{m} -antimultiplicative.*

Proof. The associativity of the product follows from braided-multiplicativity of ϕ , property (B.8) and the following interesting identities

$$(B.21) \quad g_\wedge(m \otimes \text{id}) = g_\wedge(\text{id} \otimes g_\wedge \otimes \text{id})(\text{id} \otimes \phi)$$

$$(B.22) \quad g_\wedge(\text{id} \otimes m) = g_\wedge(\text{id} \otimes g_\wedge \otimes \text{id})(\phi \otimes \text{id}).$$

The fact that 1 is the \tilde{m} -unit follows from

$$(B.23) \quad g_\wedge(1, \alpha) = g_\wedge(\alpha, 1) = \epsilon(\alpha).$$

Finally the \tilde{m} -antimultiplicativity of $*$ follows from standard commutation relations between $*$ and m, ϕ, g_\wedge . \square

Definition 7. The constructed $*$ -algebra $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ is called the *braided Clifford algebra* associated to \mathbb{V}_Σ , σ and g .

We are now going to talk about possible C^* -type norms on $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$. For this to work, it will be necessary to introduce a last set of our assumptions, regarding a more subtle behavior of σ .

A nice way to get such properties is to postulate the existence of an appropriate *auxiliary braid operator* $\tau: \mathbb{V}_\Sigma^{\otimes 2} \rightarrow \mathbb{V}_\Sigma^{\otimes 2}$, as it will be discussed near the end of this Appendix. Playing with two braid operators will also enable us to prove interesting properties of σ and its braided exterior algebra \mathbb{V}_Σ^\wedge .

We shall proceed without making any extra assumptions on the existence and properties of τ , however we have to postulate the positivity of braided antisymmetrizer maps.

(viii) **Positivity of braided antisymmetrizers.** All braided antisymmetrizer maps $A_\sigma^n: \mathbb{V}_\Sigma^{\otimes n} \rightarrow \mathbb{V}_\Sigma^{\otimes n}$ are positive operators.

The positivity property is crucial to define a C^* -algebraic structure on the Clifford algebra, because only in this case the scalar product $\langle \rangle_\wedge$ on \mathbb{V}_Σ^\wedge given by the formula

$$(B.24) \quad \langle \alpha, \beta \rangle_\wedge = g_\wedge(\alpha^*, \beta)$$

will be strictly positive (giving us a structure of a generally unbounded unitary bimodule over Σ).

Proposition B.5. *Let us consider the counit map $\epsilon: \text{cl}[\mathbb{V}, g, \sigma, \Sigma] \rightarrow \Sigma$. It is hermitian, Σ -bilinear and strictly positive.*

Proof. Hermiticity and Σ -linearity are obvious (the counit here is just the projection on Σ). The strict positivity follows from the identity

$$(B.25) \quad \epsilon(\alpha^* \beta) = \langle \alpha, \beta \rangle_\wedge. \quad \square$$

When dealing with Hilbert space operators, there is an interesting assumption we can add to the list of properties of Σ —we can assume that the set of C^* -algebraic norms on Σ distinguishes elements of Σ . Not every $*$ -algebra possesses this property, and many $*$ -algebras do not admit any representation by bounded operators. However if Σ admits C^* -algebraic norms, then they would be naturally extendible to $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$.

To see this, we can consider the *left regular representation* of $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ in the Σ -bimodule \mathbb{V}_Σ^\wedge . According to (B.25), this representation is a $*$ -representation

$$\langle \alpha, T\beta \rangle_\wedge = \langle T^* \alpha, \beta \rangle_\wedge \quad \forall \alpha, \beta \in \mathbb{V}_\Sigma^\wedge \quad \forall T \in \text{cl}[\mathbb{V}, g, \sigma, \Sigma]$$

and it is easy to see that all the operators $T \in \text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ are *continuous*. This allows us to construct natural C^* -algebra norms on $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$.

Let us now analyze a couple of interesting special cases when the positivity of antisymmetrizers would hold automatically. Assume that a selfadjoint braid operator $\tau: \mathbb{V}^{\otimes 2} \rightarrow \mathbb{V}^{\otimes 2}$ is given, satisfying $*\tau* = \tau^{-1}$, extendible by Σ -linearity to $\mathbb{V}_\Sigma^{\otimes 2}$, and such that

$$(B.26) \quad \text{im}(I - \sigma) = \ker(I + \tau),$$

or equivalently

$$(B.27) \quad \text{im}(I + \tau) = \ker(I - \sigma).$$

An immediate consequence is that σ and τ commute. Moreover

$$(B.28) \quad \text{im}(A_\sigma^n) \subseteq \left\{ \tau\text{-antisymmetric } n\text{-tensors} \right\}.$$

This inclusion is a simple consequence of (B.26) and the fact that we can write

$$A_\sigma^n = [\text{id}^k \otimes (I - \sigma) \otimes \text{id}^{n-k-2}] T_k$$

where $T_k: \mathbb{V}_\Sigma^{\otimes n} \rightarrow \mathbb{V}_\Sigma^{\otimes n}$ is the part of the antisymmetrizer sum, containing permutations whose inverse does not reverse the order of $k+1$ and $k+2$.

Proposition B.6. *Let us assume that all σ -twists act in the same way on the vectors from the space of τ -antisymmetric n -tensors. Then, this space is σ -invariant.*

If in addition $1 \in \mathbb{C}$ is the only positive eigenvalue of σ , then all braided σ -antisymmetrizers will be positive.

Proof. The fact that σ -twists act in the same way on the τ -antisymmetric vectors means that we can always (trivial for $n > 4$ for $n = 2, 3, 4$ it is necessary to use the fact that σ and τ commute) exchange them with τ -twists (acting as -1). Hence, the space of τ -antisymmetric tensors is σ -invariant.

The second assumption means that $\sigma: \ker(I + \tau) \rightarrow \ker(I + \tau)$ is strictly negative, as the space $\ker(I + \tau)$ is spanned by all negative-eigensubspaces of σ .

Therefore, all σ -antisymmetrizers are positive, and their images coincide with τ -antisymmetric spaces. \square

Proposition B.7. *Let us consider mutually equivalent properties*

$$(B.29) \quad \begin{aligned} (\tau \otimes \text{id})(\text{id} \otimes \tau)(\sigma \otimes \text{id}) &= (\text{id} \otimes \sigma)(\tau \otimes \text{id})(\text{id} \otimes \tau) \\ (\sigma \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id}) &= (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \sigma). \end{aligned}$$

*Above equations transform one to another by the *-conjugation. If they hold, and if τ -antisymmetric n -tensors are invariant under σ -twists, then all σ -twists act in the same way in this space.*

Proof. Let us consider the case $n = 3$. If $\psi \in \mathbb{V}^{\otimes 3}$ is completely τ -antisymmetric and if the invariance property holds then (B.29) gives $(\text{id} \otimes \sigma)(\psi) = (\sigma \otimes \text{id})(\psi)$. \square

Furthermore, let us observe that the following strange equalities are equivalent

$$(B.30) \quad \begin{aligned} (\tau \otimes \text{id})(\text{id} \otimes \sigma)(\tau \otimes \text{id}) &= (\text{id} \otimes \sigma)(\tau \otimes \text{id})(\text{id} \otimes \tau) \\ (\tau \otimes \text{id})(\text{id} \otimes \sigma)(\tau \otimes \text{id}) &= (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \sigma), \end{aligned}$$

as well as the equalities

$$(B.31) \quad \begin{aligned} (\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes \tau) &= (\sigma \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id}) \\ (\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes \tau) &= (\tau \otimes \text{id})(\text{id} \otimes \tau)(\sigma \otimes \text{id}). \end{aligned}$$

Actually, the equivalent equalities are just mutually adjoint. The equivalence also follows from the braid equation for τ and the fact that σ commutes with τ .

Lemma B.8. *If the above equalities hold, then the spaces of fully τ -antisymmetric tensors of order $n \geq 2$ are invariant under actions of all possible σ -twists.*

Proof. The invariance under σ -twists easily follows from the commutation property, and equalities (B.30)–(B.31). \square

Proposition B.9. *If the spaces of τ -antisymmetric operators are invariant under all possible σ -twists and if the restriction $\sigma: \ker(I + \tau) \rightarrow \ker(I + \tau)$ is negative, then all antisymmetrizer maps A_σ^n are positive.*

Furthermore, we have

$$(B.32) \quad \text{im}(A_\sigma^n) = \left\{ \tau\text{-antisymmetric tensors} \right\}.$$

for each $n \geq 2$.

Proof. At first, let us observe that braided antisymmetrizers satisfy the following interesting recursive relations

$$(B.33) \quad A_\sigma^{n+1} = \text{id} \otimes A_\sigma^n - (\text{id} \otimes Y_n)(\sigma \otimes A_\sigma^{n-1})(\text{id} \otimes Y_n^\dagger),$$

where $Y_n: \mathbb{V}_\Sigma^{\otimes n} \rightarrow \mathbb{V}_\Sigma^{\otimes n}$ is given by

$$Y_n = - \sum_{k=2}^n (-)^k \pi_{kn, \sigma}$$

and sum goes over permutations $\pi_{kn} \in S_n$ transposing 1 and the blocks $\{2, \dots, k\}$ while acting trivially in $\{k+1, \dots, n\}$.

Now, using induction on n , the negativity assumption for $\sigma : \ker(I + \tau) \rightarrow \ker(I + \tau)$, and recursive formulas (B.33) it follows that restricted antisymmetrizers

$$A_\sigma^n : \left\{ \tau\text{-antisymmetric } n\text{-tensors} \right\} \rightarrow \left\{ \tau\text{-antisymmetric } n\text{-tensors} \right\}$$

are *strictly positive, in particular invertible* operators. In particular (B.32) holds and obviously A_σ^n are positive everywhere. \square

By the way, it is worth mentioning that

$$(B.34) \quad \left\{ \tau\text{-antisymm } n\text{-tensors} \right\}^\perp = \sum_k \mathbb{V}_\Sigma^{\otimes k} \otimes \text{im}(I + \tau) \otimes \mathbb{V}_\Sigma^{n-k-2} = \ker(A_\sigma^n).$$

If (B.32) holds, it follows that the algebra \mathbb{V}_Σ^\wedge is quadratic (generated by its quadratic relations).

In this case, the Clifford algebra $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ can be viewed as the algebra built over \mathbb{V}_Σ together with generating relations

$$(B.35) \quad \sum_\alpha x_\alpha y_\alpha = \sum_\alpha g(x_\alpha, y_\alpha)$$

where $\sum_\alpha \sigma(x_\alpha \otimes y_\alpha) = \sum_\alpha x_\alpha \otimes y_\alpha$.

Well, now we can *introduce spinors* simply as vectors of irreducible representations of the *-algebra $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ in a finite-dimensional Hilbert space (or more generally, by bounded operators). Recall that every such a representation is (as generally for C*-algebras) obtained from a pure state $\omega : \text{cl}[\mathbb{V}, g, \sigma, \Sigma] \rightarrow \mathbb{C}$ via the GNS construction.

In contrast to the classical theory, the algebra $\text{cl}[\mathbb{V}, g, \sigma, \Sigma]$ may be infinite dimensional (the most interesting situations appear when Σ is infinite-dimensional) and possess non-equivalent irreducible representations.

In our main context of framed quantum principal bundles, the operator τ was coming from the appropriate bicovariant bimodule. In this context it is natural to assume that Σ is of a ‘bicovariant nature’ too. Specifically, this requires the existence of a right \mathcal{A} -module structure $\circ : \Sigma \otimes \mathcal{A} \rightarrow \Sigma$ and a right \mathcal{A} -comodule structure $\varkappa_\Sigma : \Sigma \rightarrow \Sigma \otimes \mathcal{A}$ which is a (continuous) unital *-homomorphism and such that

$$\begin{aligned} (\alpha\beta) \circ a &= (\alpha \circ a^{(1)}) (\beta \circ a^{(2)}) & 1 \circ a &= \epsilon(a) 1 \\ \Sigma(q \circ a) &= \sum_\alpha (q_\alpha \circ a^{(2)}) \otimes \kappa(a^{(1)}) c_\alpha a^{(3)} \end{aligned}$$

The maps \circ and \varkappa_Σ are completely fixed by postulating

$$\begin{aligned} g(x, y) \circ a &= g(x \circ a^{(1)}, y \circ a^{(2)}) \\ \varkappa_\Sigma g(x, y) &= (g \otimes \text{id}) \varkappa(x \otimes y). \end{aligned}$$

The above formulas extend to the elements from \mathbb{V}^\wedge straightforwardly.

APPENDIX C. INTERTWINER BIMODULES & VECTOR BUNDLES

The aim of this appendix is to sketch basic ideas and constructions related to associated vector bundles. We shall mainly follow the exposition of [9].

Let $R(G)$ be the category of finite-dimensional representations of a compact quantum group G . The objects of $R(G)$ are finite-dimensional representations of G and the arrows are the intertwiners between the corresponding representations. For each $u \in R(G)$ let us denote by H_u the corresponding carrier vector space.

Let us consider an arbitrary quantum principal G -bundle $P = (\mathcal{B}, i, F)$ over a quantum space M . Let us denote by $\mathcal{E}_u = \text{Mor}(u, F)$ be the space of intertwiners between a given representation $u: H_u \rightarrow H_u \otimes \mathcal{A}$ and $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$. The spaces \mathcal{E}_u are \mathcal{V} -bimodules, in a natural manner. We have $\mathcal{E}_\varnothing \leftrightarrow \mathcal{V}$.

In accordance with classical geometry, the spaces \mathcal{E}_u are interpretable as (the smooth sections of) the *associated vector bundles*. It is possible to give an important alternative interpretation of \mathcal{E}_u as certain invariant subspaces. More precisely, let us consider the contragradient representation $u^c: H_u^* \rightarrow H_u^* \otimes \mathcal{A}$. Then we have a natural identification

$$(C.1) \quad \mathcal{E}_u \leftrightarrow \left\{ \psi \in \mathcal{B} \otimes H_u^* \mid (F \times u^c)(\psi) = \psi \otimes 1 \right\}.$$

Explicitly, the identification is given by

$$(C.2) \quad \begin{aligned} \mathcal{E}_u \ni \psi &\longmapsto \sum_i \psi(e_i) \otimes e_i^* \\ \sum_\alpha b_\alpha \otimes f_\alpha &\longmapsto \left\{ x \mapsto \sum_\alpha b_\alpha f_\alpha(x) \right\}, \end{aligned}$$

where e_i form a basis (say, orthonormal) in H_u and $e_i^* \in H_u^*$ are the corresponding biorthogonal vectors. In various considerations it comes very handy to pass from one interpretation to another.

The following natural bimodule isomorphism holds:

$$(C.3) \quad \mathcal{E}_{u \times v} \leftrightarrow \mathcal{E}_u \otimes_{\mathcal{V}} \mathcal{E}_v \quad \forall u, v \in R(G).$$

This isomorphism is induced by the product in \mathcal{B} , via $\varphi \otimes \psi: x \otimes y \mapsto \varphi(x)\psi(y)$. Furthermore, every map $f \in \text{Mor}(u, v)$ induces, via the composition of intertwiners, a bimodule homomorphism $f_*: \mathcal{E}_v \rightarrow \mathcal{E}_u$.

We have a system of natural bimodule anti-isomorphisms $*_u: \mathcal{E}_u \rightarrow \mathcal{E}_{\bar{u}}$, defined by the diagram

$$(C.4) \quad \begin{array}{ccc} H_u & \xrightarrow{\varphi} & \mathcal{B} \\ j_u \downarrow & & \downarrow * \\ H_u^* & \xrightarrow[*_u \varphi]{} & \mathcal{B} \end{array}$$

In such a way we can interpret $\mathcal{E}_{\bar{u}}$ as the *conjugate bimodule* of \mathcal{E}_u .

Let us now focus on the main context of this paper, and consider a graded $*$ -algebra \mathfrak{hor}_P playing the role of abstract ‘horizontal forms’. Here we shall assume that $\mathfrak{hor}_P^0 = \mathcal{B}$ and that there exists a coassociative counital $*$ -homomorphism $F^\wedge: \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P \otimes \mathcal{A}$ extending the map F . Let Ω_M be the corresponding F^\wedge -fixed point subalgebra.

Applying similar intertwiner considerations to the algebra \mathfrak{hor}_P leads to Ω_M -bimodules $\mathcal{F}_u = \text{Mor}(u, F^\wedge)$. These spaces are naturally graded, and obviously $\mathcal{F}_u^0 = \mathcal{E}_u$. Moreover, it can be shown that canonically

$$(C.5) \quad \mathcal{E}_u \otimes_{\mathcal{V}} \Omega_M \leftrightarrow \mathcal{F}_u \leftrightarrow \Omega_M \otimes_{\mathcal{V}} \mathcal{E}_u.$$

These decompositions are induced by the bimodule product in \mathcal{F}_u . The spaces \mathcal{F}_u are alternatively viewable in the same way (C.2).

The structure of \mathcal{F}_u is expressible in terms of \mathcal{E}_u . Composing the above two identifications we obtain canonical flip-over maps

$$\sigma_u : \mathcal{E}_u \otimes_{\mathcal{V}} \Omega_M \rightarrow \Omega_M \otimes_{\mathcal{V}} \mathcal{E}_u.$$

These maps are grade-preserving and act as identity on \mathcal{E}_u . The Ω_M -bimodule structure on \mathcal{F}_u is expressed by the following equation

$$(C.6) \quad \sigma_u(\text{id} \otimes m_*) = (m_* \otimes \text{id})(\text{id} \otimes \sigma_u)(\sigma_u \otimes \text{id}),$$

where m_* is the product in Ω_M . Intertwiner homomorphisms f_* and conjugation maps $*_u$ obey the following diagrams

$$\begin{array}{ccc} \mathcal{E}_v \otimes_{\mathcal{V}} \Omega_M & \xrightarrow{\sigma_v} & \Omega_M \otimes_{\mathcal{V}} \mathcal{E}_v \\ f_* \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes f_* \\ \mathcal{E}_u \otimes_{\mathcal{V}} \Omega_M & \xrightarrow{\sigma_u} & \Omega_M \otimes_{\mathcal{V}} \mathcal{E}_u \end{array} \quad \begin{array}{ccc} \mathcal{E}_u \otimes_{\mathcal{V}} \Omega_M & \xrightarrow{\sigma_u} & \Omega_M \otimes_{\mathcal{V}} \mathcal{E}_u \\ *_u \downarrow & & \downarrow *_u \\ \Omega_M \otimes_{\mathcal{V}} \mathcal{E}_{\bar{u}} & \xleftarrow{\sigma_{\bar{u}}} & \mathcal{E}_{\bar{u}} \otimes_{\mathcal{V}} \Omega_M \end{array}$$

In fact, the first diagram characterizes elements of $\text{hom}(\mathcal{E}_v, \mathcal{E}_u)$ that are extendible to corresponding Ω_M -bimodule homomorphisms (the left and right Ω_M -linear extensions coincide).

We have the following natural decomposition of the algebra of horizontal forms

$$(C.7) \quad \mathfrak{hor}_P = \sum_{\alpha \in \mathcal{T}}^{\oplus} \mathcal{H}^\alpha(P) \quad \mathcal{H}^\alpha(P) = \mathcal{F}_\alpha \otimes H_\alpha$$

$$\mathfrak{hor}_P \leftrightarrow \Omega_M \otimes_{\mathcal{V}} \mathcal{B} \leftrightarrow \mathcal{B} \otimes_{\mathcal{V}} \Omega_M.$$

Let us assume that \mathfrak{hor}_P is equipped with an F^\wedge -invariant scalar product \langle, \rangle , ensuring that the spaces \mathcal{F}_α are mutually orthogonal. Then it is possible to naturally induce scalar products in all intertwiner Ω_M -bimodules. The induced product is given by

$$(C.8) \quad \langle \varphi, \psi \rangle = \sum_{ij} [C_u]_{ji} \langle \varphi(e_i), \psi(e_j) \rangle$$

where e_i form an orthonormal basis in H_u . In the alternative picture, the scalar product is simply given by tensoring the scalar products in \mathfrak{hor}_P and H_u^* . In the case of quantum Riemannian/spin manifolds, the above scalar product reads

$$(C.9) \quad \langle \varphi, \psi \rangle = \int_M \left\{ \sum_{ij} [C_u]_{ji} \varphi(e_i)^* \star_P [\psi(e_j)] \right\}.$$

Let us suppose that we have a linear operator $T : \mathfrak{hor}_P \rightarrow \mathfrak{hor}_P$ which intertwines the right action F^\wedge . Then it acts naturally in all intertwiner bimodules \mathcal{F}_u . The action is defined by simply taking the composition with the intertwiners. In other words

$$[T_u \psi](x) = T[\psi(x)] \quad \psi \in \mathcal{F}_u, x \in H_u$$

or in the alternative picture

$$T_u \left\{ \sum_{\alpha} \varphi_{\alpha} \otimes f_{\alpha} \right\} = \sum_{\alpha} T(\varphi_{\alpha}) \otimes x_{\alpha}.$$

It is clear that in such a way all algebraic relations between F^{\wedge} -covariant operators in \mathfrak{hot}_P are preserved. Furthermore, the adjoint operation is preserved, in a natural manner. More precisely, if T is formally adjointable then T_u is formally adjointable too, for each $u \in R(G)$ and $[T_u]^{\dagger} = [T^{\dagger}]_u$. This is a direct consequence of the definition of the scalar product in our bimodules.

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